CENTRALIZED AND DECENTRALIZED MANAGEMENT OF WATER RESOURCES WITH MULTIPLE USERS

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DOCTOR OF PHILOSOPHY

By

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June, 2011

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

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ABSTRACT

CENTRALIZED AND DECENTRALIZED MANAGEMENT OF WATER RESOURCES WITH MULTIPLE USERS

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June, 2011

In this study, we investigate two water inventory management schemes with multiple users in a dynamic game-theoretic structure over a two-period planning horizon. We first investigate the groundwater inventory management problem (i) under the decentralized management scheme, where each user is allowed to pump water from a common aquifer making usage decisions individually in a non-cooperative fashion, and (ii) under the centralized management scheme, where users are allowed to pump water from a common aquifer with the supervision of a social planner. We consider the case of n non-identical users distributed over a common aquifer region. Furthermore, we consider different geometric configurations overlying the aquifer, namely, the strip, ring, double-layer ring, multi-layer ring and grid configurations. In each configuration, general analytical results of the optimal groundwater usage are obtained and numerical examples are discussed. We then consider the surface and groundwater conjunctive use management problem with two non-identical users in a dynamic game-theoretic structure over a planning horizon of two periods. Optimal water allocation and usage policies are obtained for each user in each period under the decentralized and centralized settings. Some pertinent hypothetical numerical examples are also provided.

Keywords: Groundwater, Surface Water, Centralized and Decentralized Management, Conjunctive Water Use, Darcy's Law, Nash Equilibrium.

ÖZET

MERKEZİ VE MERKEZİ OLMAYAN SU KAYNAKLARININ ÇOK KULLANICILI YÖNETİMİ

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Haziran, 2011

Bu çalışmada iki periyotluk dinamik oyun teorisi yapısında çok kullanıcılı suyun envanter yönetimini iki farklı durum için gözlemledik. Özellikle, ilk olarak her bir kullanıcılı ortak bir akiferden su pompalayabildiği, kararlarını bireysel olarak kooperatif olmayacak şekilde aldıkları (i) merkezi olmayan yeraltı suyu yönetimi problemini, (ii) merkezi bir planla bir sosyal plancının önerisiyle kullanıcıların su pompaladıkları durumları inceledik ve n tane birbirinden farklı kullanıcının ortak bir akifer alanına dağıtıldığı durumu gözlemledik. Ayrıca, akiferin üzerini örten şerit, halka, iki katmanlı halka, çok katmanlı halka ve ızgara gibi değişik geometrik konfigürasyonları inceledik. Her bir konfigürasyonda, optimal yeraltı suyu kullanımıyla ilgili analitik sonuçlar elde edildi ve merkezi ve merkezi olmayan durumlar için sayısal örnekler verildi. Sonrasında, yüzey ve yeraltı suları yönetimini birlikte değerlendirdiğimiz birbirinden farklı iki kullanıcı için dinamik oyun teorisi yapısında iki periyotluk problemi göz önünde bulundurduk. Yani, merkezi olmayan yönetim planlaması için, her bir kullanıcı her iki su kaynağından da kooperatif olmayan şekilde su kullanabilmektedir. Öte yandan, merkezi yönetim planlamasında ise, iki kullanıcı da her iki su kaynağını bir sosyal planlamacının gözetimi altında kullanabilmektedir. Optimal su paylaşımı ve kullanım politikaları her bir kullanıcı için merkezi ve merkezi olmayan durumlarda elde edilmiştir. Bazı konuyla alakalı varsayımlı sayısal örnekler de ayrıca verilmiştir.

Anahtar sözcükler: Yeraltı Suyu, Yüzey Suyu, Merkezi ve Merkezi Olmayan Yönetim, Birbirine Bağlı Su Kullanımı, Darcy Yasası, Nash Dengesi.

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To the memory of my father ...

Contents

List of Figures

List of Tables

Chapter 1

Introduction

Effective management of limited resources shared by multiple users is becoming of more importance due to increasing pressures resulting from demographic and/or economic growth and ecological deterioration. Such resources include fisheries, water and clean air. These resources suffer from either lack of enforceable private property rights or their designation of common/public property. Furthermore, they exhibit an interesting property; they tend to move from one location to another depending on the extent of usage. Water is a vital source for sustainability and efficient use of it is essential for life on earth. Underground water laterally flows within an aquifer along with the hydrological gradient (difference between low and high water levels) as governed by Darcy's Law; schools of fish travel to other locations to run away from heavy fishing in one location; pollution at a point is dissipated degrading the overall quality over a larger area. This property permits gaming behavior among users upon using of these resources. In spite of the fact that about two-thirds of the earth's surface is covered by ocean water, fresh water supplies are becoming more limited and scarce due to the continuous growth of population, particularly in developing countries. Fresh water supplies may come from surface water bodies like rivers and lakes or from groundwater. The availability of surface water depends on the annual quantities of rainfalls and water harvesting collected and stored in main reservoirs. Groundwater is the water that has percolated to a usable aquifer that provides water storage. Scarcity of water - for personal and industrial/agricultural use - is increasing in both absolute and relative terms. Shortages observed in rainfall, adverse micro-climatic changes, contamination of groundwater reservoirs (aquifers) due to increasing industrial and human pollution result in a decrease in the amount of water of certain quality fit for use. Increases in demand for water due to growth in the overall populations and changes in consumption patterns result in the relative scarcity of this precious resource. In arid and semi-arid regions of the globe, the scarcity is reaching critical levels. The gaming behavior of users may be detrimental for many communities for some generations to come. In the context of fresh water resources usage management, users represent water users in the macro as well as in the micro levels of real-life applications. In the macro level, users might represent two neighbor countries, each has its own sources of surface water (main and local reservoirs), and simultaneously shares the stock of a transboundary groundwater aquifer (basin) with its neighbor. Each country aims at determining the optimal polices of water usage from both sources taking into account the commonality of groundwater stock with the other country in order to maximize its water usage profits realized over time. This fierce competition on common groundwater stocks might result in unfair allocation of this valuable resource between these countries and sometimes it might result in serious political crises and conflicts. Another real application arises in two neighboring cities, where one city has its own surface water sources and uses water chiefly for industrial purposes and, simultaneously, shares a common groundwater stock with an adjacent city having its own surface water sources as well but mainly consumes water for residential purposes (drinking). Industrial consumers accelerate the depletion of the common stock of groundwater on the account of urban users who might suffer frequent deprivation of drinking water as a result of that. Many applications might be visible in many micro level in reality. One application of that is when users represent different industries, each having its own stock of surface water stored in its own reservoirs and shares the common groundwater stock with an adjacent industry. Industries might be non-identical because their water usage cost and their water usage revenue structures might be different. In the sequence, one industry might face several water shortages due to the unfair usage of the common groundwater stock of its rival. Another application, like the one of this study, where fresh water supplies are used for agricultural purposes to irrigate different crops. In this case, users represent adjacent farms in which different crops with different yields are irrigated by farm owners (farmers) under different water usage (holding and pumping) technologies .

In this study, we investigate the management problem of water usage and allocation among multiple users under dynamic game-theoretic structures. More formally, in the first part of this study, we investigate two groundwater inventory management schemes with multiple users in a dynamic game-theoretic structure: *(i)* under the decentralized management scheme (decentralized problem), each user is allowed to pump water from a common aquifer making usage decisions individually in a non-cooperative fashion. Under this setting, each user's objective is to choose the water usage quantity that maximizes her own profit realized from water usage taking into account the usage quantities (responses) of her neighbors. On the other hand, *(ii)* under the centralized management scheme (centralized problem), users are allowed to pump water from a common aquifer with the supervision of a social planner, who is interested in determining the water usage quantities of all users which maximize the total water usage realized profits. This work is motivated by the work of Saak and Peterson [52], which considers a model with two identical users sharing a common aquifer over a two-period planning horizon. Groundwater is pumped from the aquifer and used for agricultural purposes to satisfy the irrigation demands of some growing crops in some agricultural areas. In this work, the model and results of Saak and Peterson [52] are generalized in several directions. Specifically, we first build on and extend their work to the case of n non-identical users distributed over a common aquifer region. Furthermore, we consider different geometric configurations of users overlying the aquifer, namely, strip, ring, double-layer ring, multi-layer ring and grid configurations. In each configuration, general analytical results of the optimal Nash equilibria of groundwater usage are obtained in decentralized problems. Besides, the optimal equilibrium water usage quantities are obtained in the centralized problem. We also show that the coordination of the decentralized and centralized solutions can not be achieved through a simple pricing mechanism.

In order to restrict the withdrawal of the limited quantities of groundwater,

water authorities allow conjunctive use of surface and groundwater to meet users demands with the aim of minimizing the undesirable physical, environmental and economical effects of individual source usage and optimizing the water demand/supply balance. Conjunctive use is usually considered as a reservoir management program, where both surface water reservoirs and groundwater aquifer belong to the same basin. In reality, users receive surface water from an external source (main reservoir) and keep their stocks in their own reservoirs while, at the same time, they overlay and share common groundwater stocks stored in underground aquifers. Users who might differ in their water demand requirements as well as in their water usage revenue-cost structures, apply the conjunctive use of surface and groundwater over time in order to maximize their water usage benefits. The term users represents water users on the macro as well as on the micro levels of real-life applications. In one application, like the one of our work, conjunctive use is practiced for agricultural purposes to irrigate different crops, where users represent adjacent farmers who plant and irrigate different crops under different revenue (crop yield) and water usage cost (holding and pumpage) structures. In the second part of this study, we investigate the conjunctive water use management problem with two non-identical adjacent users in a dynamic game-theoretic setting under two management schemes. Namely, *(i)* Decentralized management scheme (decentralized problem): In this setting, each user is allowed to use surface and groundwater, respectively, from her reservoir and from the common aquifer making water usage decisions individually in a non-cooperative fashion. For a given response (usage quantities) of her neighbor, each user is interested in determining her optimal operating policy that maximizes her total discounted profit realized from water use over a two-period planning horizon. As users share a common groundwater aquifer, upon pumpage of groundwater, water starts to transmit laterally between them in accordance with Darcy's Law. In the sequel, users compete and behave greedily in order to use as much groundwater as possible. This greedy behavior creates a non-cooperative form-game between users. *(ii)* Centralized management scheme (centralized problem): Here, users are allowed to use surface and groundwater from their own reservoirs and from the groundwater aquifer, respectively, with the supervision of a social planner (water authorities). The social planner is interested in the efficient utilization and allocation of the limited quantities of surface and groundwater among users. The problem is to determine an optimal operating policy of the water system that maximizes the total discounted profits realized from both surface and groundwater usage in a two-period planning horizon. Such an operating policy identifies, at each period, for each user, the quantity of surface water released as well as the quantity of groundwater pumped, both quantities are consumed to satisfy irrigation demands.

This work is motivated by the work of Saak and Peterson [52] as well as by earlier works on conjunctive use management (Noel et al. [47], Azaiez and Hariga [7], Azaiez [6] and Azaiez et al.[8]) to consider a more comprehensive and more realistic model in reality. Our model incorporates the conjunctive use of ground and surface water in one setting that permits the sharing of groundwater aquifer. This commonality of groundwater results in a game-theoretic dynamic structure among users who use surface water in conjunction with groundwater to satisfy their irrigation demands. Users acquire their private surface water stocks from an external supplier (external reservoir) and keep them at their own local reservoirs to be used conjunctively with groundwater. We study the above-mentioned conjunctive water use management problems with two non-identical users in a dynamic game-theoretic structure over a planning horizon of two periods. Under the decentralized problem, optimal water allocation policies and general Nash equilibria are obtained for each user in each period. Additionally, for the special case of identical users, Nash equilibria are found to be symmetric. Optimal water allocation polices as well as equilibrium water usage, for each user in each period, are also obtained under the centralized problem. Besides, for the special case of identical users, unique, symmetric and groundwater aquifer's transmissivity-independent solutions are found. Our analytical results also reveal the possibility of coordinating the two solutions through achieving the centralized solution in the decentralized problem when users are identical.

We begin with a review on the relevant literature of this study in Chapter 2. We first present the literature pertinent to the first part of this study; centralized and decentralized management of groundwater with multiple users. Then, we present the literature related to the second part of the study; the centralized and decentralized management of conjunctive use of surface and groundwater.

In Chapter 3, we define the problem of the centralized and decentralized management of groundwater with multiple users. We present the preliminaries and the specifics of the model and the analytical results of the two water management schemes for different geometric configurations. We show the existence of a unique Nash equilibrium and provide the solution structure for the decentralized problems with n non-identical users. For identical users, we also manage to derive explicit solutions for the optimal water usage. It is shown that in strip configuration with n identical users, the optimal Nash equilibrium usage quantities oscillate about the optimal Nash equilibrium usage quantities of the ring configuration. The analysis for the centralized problem reveals that the optimal solution of groundwater usage is symmetric, unique across users and independent of the characteristics of the groundwater aquifer. This generalizes one of the important findings of Saak and Peterson [52] regarding the optimal equilibrium water usage. An important question that might be raised by a policy maker is about the possibility of coordinating the groundwater system by achieving the centralized solution in the decentralized game theoretic setting via a single pricing mechanism. Our results show that this is not possible to be realized. We also consider a general extension to our work. Namely, for both strip and ring configurations, we investigate the water management problems for a model with a salvage value function, where part of water stock in the second period is allowed to partially satisfy crops' irrigation demands. The related analytical results for the new model are also presented

Chapter 4 presents the results of a numerical study which has been conducted for various number of users to compare water usages and the resulting profits under the decentralized and centralized problems. The results are presented and compared for all configurations considered. In our numerical results with time-invariant parameters, we observe that, in both strip and ring configurations, as the underground water transmission coefficient increases, users become more greedy and use more water in the decentralized problem. This greedy behavior however adversely affects the system's total discounted profit. For time-variant parameters, we study the effect of changing the crop unit price and yield function parameters on the optimal solution as well as on the realized total profits in the centralized and decentralized problems. In all settings (variant and invariant), as expected, the centralized solutions always dominate the decentralized ones by achieving more profits. We note that although the optimal solutions of the strip structure do not converge to that of the ring structure as the number of users increase, they are observed to become very close in our numerical examples for the non-extreme users of the strip. We also provide and discuss some illustrative numerical examples for the other geometric configurations; namely, double-layer, multi-layer and grid ones.

In Chapter 5, we introduce the problem of conjunctive use management of surface and groundwater for the centralized and decentralized settings. We present a detailed description of the model, the main assumptions and some structural properties of water usage profit function. We also discuss the analytical solutions of the decentralized and centralized problems. Optimal water allocation policies and general Nash equilibria are obtained under the decentralized problem. Under the centralized problem, optimal water allocation polices as well as equilibrium water usage are also obtained. We also provide some illustrative numerical examples to assess the effect of the discount rate on the optimal solution for both problems for identical users having the same, but time-variant parameters, with finite and infinite transmissivity coefficients. We observe that under certain parameters setting, it is possible to coordinate the conjunctive use system by achieving the centralized solution in the decentralized problem. It is also noted that total decentralized and centralized profits turn out to increase exponentially with the discount rate.

In the last chapter, some concluding remarks about the study and future research directions are provided.

Chapter 2

Literature Review

In this chapter, we provide a review of the literature relevant to this study. In Section 2.1, a general literature on water reservoir management is introduced. Section 2.2 presents the literature on the groundwater management. In this section, the literature which is closely-pertinent to the first part of our study is provided and then the literature on general groundwater management is introduced. Section 2.3 presents the literature related to management of conjunctive use of surface and groundwater.

2.1 Literature on Water Reservoir Management

In this section, we introduce a general review of the literature concerning the operation, management, optimization and design of water reservoir system. A large part of the literature has discussed the optimization models of operations and management of single and multi-surface water reservoirs. The reader can find a full state-of-the-art review of water reservoir management and optimization models used for single- and multi- reservoir systems in Labadie [34], Lund and Gumzan [38], Yakowitz [74], and Yeh [75]. In particular, these review studies present surveys on various optimization and mathematical models and algorithms developed for reservoir operation, namely, linear and nonlinear programming, dynamic programming, simulation, stochastic programming, optimal control theory, multi-objective programming, network and heuristic programming models.

On the other hand, many studies have been devoted to the operation of water reservoir systems in drought periods utilizing hedging policies on water demand. To name only a few, we encourage the reader to refer to Lund and Reed [39], Shiau [58], Shih and Revelle [60], Shih and Revelle [61] Neelakantan and Pundeadikanthan [44], Shiau and Lee [59], Tu et al. [65], and Vasiliadis and Karamouz [67] for more details. Another part of literature is concerned with the design of single- and multiple- reservoir systems and water distribution networks, as well as with optimal expansion and installation polices of additional supply facilities. For more details about this part of literature, the reader can refer to Armstrong and Wills [4], Arunkumar and Chon [5], Babayan et al. [9], Cervellera et al. [14], Firoozi and Merrfiled [21], Lamond and Sobel [35], and Sharma et al. [57]. Many works have been devoted to study the ability of existing and proposed water supply systems to operate satisfactorily under the wide range of possible future demands. Researchers have been developing system performance criteria to capture particular aspects of possible system performance which are especially important during drought periods, peak demands or extreme weather. Important references dealing with water supply system performance criteria include Bayazit and Unal [11], Hashimoto et al. [29], Mondal and Wasimi [42], Moy et al. [43], Srinivasan et al. [62], Srinivasan [63] and Wang et al. [71].

Game-theoretical models have been developed and solved in reservoir optimization/operation to take into consideration the potential interactions, behavior preferences of water users, reservoir operator and their associated modeling procedures within the stochastic modeling framework as shown in Ganji et al. [22] and Ganji et al. [23]. More specifically, they utilize game theory to present the associated conflicts among different consumers due to limited water through developing stochastic dynamic game-theoretical models with perfect information about the associated randomness of reservoir operation parameters. Different solution methods including simulated-annealing approach are utilized to solve the models and the results are compared with alternative reservoir operation models, like Bayesian stochastic dynamic programming, sequential genetic algorithm and classical dynamic programming regression. Another study by Ganji et al. [24] employed a fuzzy dynamic game-theoretical models to handle the water allocation management problem in a reservoir system. A recent study by Homayoun-far et al. [30] developed and solved a continuous model of dynamic game for reservoir operation, where two solution methods are used to solve the model of continuous dynamic game.

2.2 Literature on Groundwater Management

Closely-related to this study, several studies have been devoted to the groundwater usage and allocation over time. In an early work, Burt [13] discussed the optimal allocation over time of a single resource (mineral deposits, groundwater, petroleum, wildlife and fish) utilized by a single user which is either of fixed supply or only partially renewable at a point in time. The allocation problem was formulated as a dynamic program and approximate decision rules for resource use were derived as a function of current supply, using first and second degree Taylor's series approximation. Gisser and Sanchez [25] argued that applying different groundwater management strategies result in a negligible welfare gain for practical policy considerations. More specifically, they conducted an analytical comparison between two distinct groundwater management strategies; the no control (decentralized) strategy and the optimal control (centralized) strategy. It was shown that if the aquifer's storage capacity is large enough, then the two strategies perform equally well in terms of the welfare gain from groundwater usage. Allen and Gisser [3] extended the work in [25] by considering a non linear demand function for water use. They confirmed that if water rights are properly defined and if the aquifer's storage capacity is relatively large, then the difference between no control strategy and optimal control strategy is small and thus can be ignored for practical considerations.

However, Negri [45] pointed that the assumption of openly accessed groundwater aquifer adopted in [25] and [3] is not valid for all aquifers since access to groundwater is usually limited due to the need for users to acquire the overlying land as well as the water right. Negri [45] developed a differential dynamic game-theoretic models of groundwater in a restricted access setting assuming an infinite groundwater aquifer's transmissivity. The dynamic interactions among the aquifer users are addressed by modeling the common property aquifer as a dynamic game in a continuous time. The open-loop and feedback equilibria were compared. More specifically, open-loop equilibria assume that groundwater users commit themselves in the initial time to a complete time path of water pumpage that maximizes the present value of their stream profits given the pumpage paths of their competitors. The solutions resulted from open-loop equilibria are an optimal set of path strategies (pumpage policies) for each user, where the rate of usage over time depends only on time and not on the actions of other users or on the observed water stock level. On the other hand, in the feedback equilibria, instead of selecting path strategies, usage decisions depend on time and the water stock level taking into account the actions of other users. The results showed the superiority of the feedback solution because it handles both the pumpage cost externality and the strategic externality resulting from the competition between users on groundwater stock, whereas the open-loop solution considers only the pumpage cost externality.

The previous studies by Burt [13], Gisser and Sanchez [25], Allen and Gisser [3] and Negri [45] represent a line of research in which the precise individual incentives leading to welfare losses are identified, [52]. However, Saak and Peterson [52] pointed to another line of research in which the single cell aquifer in Gisser and Sanchez's [25] model is replaced by another model which accurately depicts the groundwater hydrology. Specifically, the single cell aquifer model assumes instantaneous (infinite) lateral flow of groundwater and, hence, the water pumpage by one user has an immediate and equal impact on the water availability to other users in the system. However, in reality, this is not the case. In particular, the speed of aquifer transmissivity (lateral flow) of groundwater among users on the aquifer is slow on average and depends on a number of spatially aquifer properties, [52]. For a model with spatially distributed users over an aquifer with finite transmissivity, a general social planner's problem is studied by Brozovic et al. [12]. They found that the dynamic optimal pumpage rates change spatially across the aquifer, a result which could not be shown in Gisser and Sanchez's [25] model. Nevertheless, Brozovic et al. [12] did not study the common property equilibrium in their setting.

Saak and Peterson [52] built on the game-theoretic and spatial groundwater aquifer models to investigate the the effect of incomplete information about the aquifer transmissivity on the common property equilibrium. They argue that although users know the dependence of their water stock availability on the extraction activities of their neighbors on the aquifer, they do not know the degree of these activities accurately. Furthermore, aquifer transmissivity data at certain locations on the aquifer are limited and difficult to be inferred from the water stock levels and extraction rates at neighboring locations as these rates are private information, [52]. To study the effect of this information issue, Saak and Peterson [52] developed a game-theoretic model with restricted aquifer access, where water usage at one location impacts the future water stocks at neighboring locations depending on the unknown aquifer transmissivity. Infinite transmissivity of the aquifer represents an extreme case in which the aquifer consists of independent cells with zero lateral flows. Saak and Peterson [52] considers the simplest setting in their model composed of two identical users sharing and using the groundwater aquifer over a finite planning horizon of two periods. Their contribution is two-fold: they model underground hydrological behavior more realistically and they incorporate the possibility of lack of information about the ground transmissivity by users. Their analysis revealed that better information may lead to either increase or decease in the equilibrium extraction rate, which in turn may lead to either increase or decrease in equilibrium welfare. Also, they showed that with better information about the aquifer transmissivity, the extraction rate gets closer to the centralized (social planner's) solution, however, the welfare decreases. Furthermore, they pointed that the curvature property (concavity) of users' water usage net benefit functions has a crucial role in determining the direction (increase/decrease) of impact realized from better information.

The model of Saak and Peterson [52] is restricted to two identical users and two periods. They argued that the extension of their model to a multi-cell (user) aquifer may result in different usage quantities even when users are identical. The multi-period setting, as argued by Saak and Peterson [52], is more complicated to be addressed since information about the aquifer transmissivity impacts both the speed of extraction and the lifetime of the aquifer, even for rechargeable aquifers. More specifically, when users are better informed about the aquifer transmissivity, the lifetime of the aquifer may increase or decrease depending on the properties of the water benefit function and the periodic discount rate. In the next chapter, we discuss the analysis of the extension of Saak and Peterson's [52] model to the case of multiple non-identical users for a two-period planning horizon. In particular, we study the groundwater aquifer's management problem under the centralized (social planner) and the decentralized management schemes for different geometrical configurations of user overlaying the aquifer. Nevertheless, our scope is not on the transmissivity's information issue, we develop and analyze our model over a finite planning horizon of two periods due to the justification adopted by Saak and Peterson [52].

The literature on general groundwater management is considerably rich. One important bulk of the literature has been devoted to developing and solving optimization models of groundwater management, including but not limited to Aguado and Remson [2], Remson and Gorelick [50], Wanakule and Mays [70], Willis and Liu [72], Willis and Newman [73], Haouari and Azaiez [28], Qureshi et al. [49], Stoecker et al. [64]. Simulation combined with optimization has been extensively used in groundwater management resulting in the so-called simulation-management models (see Wanakule and Mays [70], Mc Phee and Yeh [41], and Usul and Balkaya [66]). The study by Gorelick [26] presents a review of the literature of such models. Due to commonality of groundwater, another line of research employs game-theoretical models to handle the water usage and allocation conflicts among parties involved in the system. To name but a few, we have Negri [45], Saak and Peterson [52], Chermak et al. [15] and Eleftheriadou and Mylopoulos [20].

2.3 Literature on Conjunctive Use Management

This section presents the literature including the studies pertinent to the conjunctive use management of surface and groundwater. In a recent work, Roberts [51] summarized the chronological development of conjunctive use from various aspects. She pointed that conjunctive use as a water strategy was discussed in early studies in fifties and sixties of the last century, while the positive and negative economic analysis of conjunctive use was discussed in some hydrology texts. Besides, she mentioned the works considering the application of optimization techniques utilized in the allocation models of agricultural areas as well as the models of design and operation of dams and groundwater aquifers in agricultural applications. Several works have been devoted to the optimization models of managing the conjunctive use of surface and groundwater under deterministic and stochastic settings. Afshar et al. [1] developed and implemented a hybrid two-stage genetic algorithm and a linear programming algorithm to optimize the design and operation of a large-scale surface water and groundwater irrigation system. They derived a set of optimal operating rules for the joint utilization of water storage capacities to meet irrigation demand requirements. Another work by Vedulaa et al. [68] is concerned with the derivation of an optimal conjunctive use policy for irrigation of multiple crops in a reservoir-canal-aquifer system. Through the objective of maximizing the total yields of crops over a year, the integration of the reservoir operation for canal release, groundwater pumpage and crop water allocations for each season was achieved.

Lu et al. [36] developed an inexact rough-interval two-stage stochastic programming (IRTSP) method for conjunctive water allocation problems. Through introducing rough intervals to the modeling framework, a conjunctive water-allocation system was structured for characterizing the proposed model. Comparisons of the proposed model to a conventional and an interval two-stage stochastic programming model implied the reliability of IRTSP method. Diaoa et al. [17] analyzed groundwater regulation in a general equilibrium setting by considering the stabilization value of groundwater under drought and rural-urban surface water transfer shocks. They evaluated the direct and indirect effects of groundwater regulation on agriculture and non agriculture sectors. Specifically, the studied the effects of an increase in groundwater pumpage cost, a transfer of surface water from rural to urban use and a reduction of water availability due to severe drought. Marques et al. [40] applied a two-stage stochastic quadratic programming to optimize conjunctive use operation of groundwater pumpage and artificial recharge with farmer's expected revenue and cropping patterns. Their results showed potential gains in expected net benefits and reduction in income variability from conjunctive use, with increase in high value permanent crops along with more efficient irrigation technology.

Other works utilized simulation models accompanied with optimization models for handling conjunctive use management. Safavi et al. [53] developed an artificial network model as simulator of surface water and groundwater interaction and a genetic algorithm as the optimization model. Their main goal was to minimize shortages in meeting irrigation demands for three irrigation systems subject to constraints on the control of the underlying water table and maximum capacity of surface water irrigations systems. Sarwar and Eggers [56] developed a conjunctive use model to evaluate alternative management options for surface and groundwater resources. The groundwater model takes net recharge as an input from the water balance calculation and simulates flow in the groundwater under all boundary stresses. A geographical information system was used to assemble various types spatial data. Ejaz and Peralta [18] developed a simulation-optimization model to address the common conflicts between water quantity and quality objectives. The quantity objective is to maximize steady conjunctive use of groundwater and surface water resources, whereas the quality objective is to maximize waste loading from a sewage treatment plant to the stream without violating some quality limits. Velazquez et al. [69] developed and integrated hydrologic-economic modeling framework for optimizing conjunctive use of surface and groundwater at the river basin scale. They simulated the dynamic stream-aquifer interaction to get a more realistic representation of conjunctive use. The associated economic results were obtained through maximizing the net value of water use. Başağaoğlu et al. [10] formulated a nonlinear coupled simulation and optimization model to determine the optimal operating policies with a minimal cost for the conjunctive management of hydraulically integrating surface and groundwater supplies. To eliminate nonlinearity, an approximating problem was formulated as linear mixed integer program and the solution was found to be in good agreement with the simulation results of the original nonlinear problem.

Knapp and Olson [32] have concentrated on the economic analysis and efficiency of conjunctive use in agricultural applications. Several studies have been focused on the optimization of conjunctive use of groundwater and surface water under different settings. Noel et al. [47] addressed an optimal control model to determine the socially optimal spatial and temporal allocation of ground and surface water between agricultural and urban uses. Another work by Azaiez [6], considered the ground and surface water conjunctive usage model for a single user over a multi-period (not more than 5 years) planning horizon, allowing for aquifer artificial recharge. The model resulted in an operating policy that determines the total amount of surface water to import and rations of that total amount to be allocated to irrigation demands and that to artificial recharge as well as the groundwater pumpage quantity at each period. The work by Azaiez and Hariga [7] considered a model with main reservoir receiving a stochastic supply of water and feeds n local reservoirs, each faces a stochastic demand over a time horizon of one period (season). In case of supply shortages, the water supply from the main reservoir to local ones is supplemented with emergency withdrawals from a groundwater aquifer. The model identifies the optimal release policy of surface water form the main reservoir and of groundwater (if any) and from local reservoirs to irrigation areas in one season such that the total profit of the region is maximized. The work by Azaiez et al. [8] extended the work by Azaiez and Hariga [7] to incorporate the case of multi-periods model.

2.4 Summary

We observe that the first part of literature on groundwater in Section 2.2 focuses on the optimization and management of a single water source (groundwater) usage and allocation among non-identical users (single and multiple) with (finite and infinite) aquifer transmissivity over discrete and continuous time horizons under dynamic game-theoretic structures. Also, we observe that the second part of literature in Section 2.3 focuses on the conjunctive use of surface and groundwater under different deterministic and stochastic settings. But, the conjunctive use models discussed in in Section 2.3 lack taking into consideration the commonality property of groundwater aquifer among users. For example, in the works by Noel et al. [47], Azaiez and Hariga [7], Azaiez [6], Azaiez et al. [8]), the groundwater aquifer is utilized by a single user and its stock is not shared with other users. In other words, there is no commonality of groundwater among multiple users. Also, the works of Azaiez et al. [8] and Azaiez and Hariga [7], considered groundwater as a standby source of water supply to supplement any water supply shortages from the main reservoir to the local ones. Therefore, being utilized by a single user, lateral transmissivity of groundwater does not exist in these works' models and, hence, non of them includes any dynamic game-theoretic setting in their structure. Furthermore, the game-theoretical models presented in Ganji et al. [22], Ganji et al. [23], Ganji et al. [24] and Homayoun-far et al. [30] in Section 2.1 addressed the optimization/operation models of a single water resource (surface water) in reservoir system within the framework of dynamic game-theoretical models. We also observe that these works lack the inclusion of another source of water supply (groundwater) in addition to the main source (surface water) in their models.

Earlier works on groundwater management and conjunctive use management and the above-mentioned observations about the two main parts of literature motivate us to consider a more comprehensive and more realistic model in reality with two non-identical users over a planning horizon of two periods. Our model incorporates the conjunctive use of ground and surface water in a setting that permits the sharing of groundwater aquifer possessing a *finite* transmissivity coefficient. This commonality of groundwater results in a game-theoretic dynamic structure among users who use their own private sources of surface water in conjunction with the common groundwater aquifer in order to satisfy their irrigation demands. Users acquire their private surface water stocks from an external supplier (external reservoir) and keep them at their own local reservoirs to be used conjunctively with groundwater.

In Chapter 5, we discuss the analysis of the conjunctive water use model. We study the conjunctive use management problem under the centralized (social planner) and the decentralized management schemes. In our analysis, we provide the optimal solutions of the problem under both management settings. The case of multi-non-identical users will be considered as a future research direction.

Chapter 3

Centralized and Decentralized Management of Groundwater With Multiple Users

In this chapter, we consider the model of multi-non-identical users, with time-variant parameters, overlying and sharing a common groundwater aquifer under a dynamic game-theoretic setting over a planning horizon of two periods. The groundwater management problem corresponding to this model is investigated from the decentralized and centralized management perspectives for different geometrical configurations of users occupying the aquifer region.

The main assumptions and basic properties of the model will be explained in Section 3.1. Section 3.2 presents the analysis of the decentralized and centralized management problems of the first geometrical configuration; the strip configuration. In Section 3.3, we present the analysis of both management problems corresponding to the second geometrical configuration; the ring configuration. Sections 3.4 and 3.5, respectively, present the analysis of both management problems in the double-layer and multi-layer ring configurations. In Section 3.6, we provide the analysis of the grid configuration. In the last section, Section 3.7, we revisit the results of the strip and ring configurations through augmenting an appropriate salvage value function in the second period of the model.

3.1 Preliminaries and Basic Model Properties

In this section, we lay out some common assumptions and model properties in our analysis. We consider a system of n non-identical users using a common groundwater aquifer, where users aim to maximize their discounted profits over a finite planning horizon of T periods. We consider both centralized and decentralized settings. Next, we introduce the notation pertinent to this chapter.

Notation

Strip and Ring Configurations

$x_{i,t}$: groundwater stock level at the beginning of period t for user i

$x_{i,0}$: initial groundwater stock level at the beginning of the planning horizon for user i; $(x_{i,0} = x_{i,1}, \forall i)$

$w_{i,1}$: aquifer recharge amount at the beginning of period two for user i

$u_{i,t}$: groundwater pumpage (and usage) quantity by user i in period t

$u^*_{i,t}$: groundwater optimal pumpage (and usage) quantity by user i in period t

α: groundwater aquifer's transmissivity (lateral flow) coefficient; $\alpha \in [0, 0.5]$

$\beta_{i,t}$: discount rate for user i in period t

β_t: discount rate for all users in period t

$\nu_{i,t}$: utility-of-income function for user i in period t

$y_{i,t}(u_{i,t})$: crop's yield function for user i in period t

$\tau_{i,t}(u_{i,t}, x_{i,t})$: groundwater extracting (pumping) cost function for user i in period t

$k_{i,t}$: fixed cost function of infra-structural (farming) inputs for user i in period t

$a_{i,t}$: output price of the crop when the crop production quantity is zero for user i in period t

$b_{i,t}$: rate of decrease in crop's output price with respect to the crop's production for user i in period t

$c_{i,t}$: unit cost of groundwater extraction (pumpage) for user i in period t

$g_{i,t}(u_{i,t}, x_{i,t})$: groundwater usage profit function for user i in period t

$Q_{i,j}$: lateral flow of groundwater in period one between users i and j; $i \neq j$

$\vec{u}_t$: groundwater usage vector for all users in period t

$\vec{x}_t$: groundwater stock vector for all users in period t

$\Gamma_{i,t}(\vec{u}_t, \vec{x}_t)$: total discounted profit from groundwater usage for user i in period t

$\Gamma^*_{i,t}(\vec{u}_t, \vec{x}_t)$: maximum total discounted profit from groundwater usage for user i in period t

$\tilde{\Gamma}_t(\vec{u}_t, \vec{x}_t)$: total discounted profit from groundwater usage for all users in period t

$\tilde{\Gamma}^*_t(\vec{u}_t, \vec{x}_t)$: maximum total discounted profit from groundwater usage for all users in period t

Double and Multi-Layer Ring Configurations

$x_{(i,k),t}$: groundwater stock level at the beginning of period t for user (i, k)

$x_{(i,k),0}$: initial groundwater stock level at the beginning of the planning horizon for user (i, k); $(x_{(i,k),0} = x_{(i,k),1}, \forall\ i, k)$

$w_{(i,k),1}$: aquifer recharge amount at the beginning of period two for user (i, k)

$u_{(i,k),t}$: groundwater pumpage (and usage) quantity by user (i, k) in period t

$u^*_{(i,k),t}$: groundwater optimal pumpage (and usage) quantity by user (i, k) in period t

$\alpha_{(i,k)}$: aquifer's lateral transmissivity coefficient between identical adjacent users $(i-1, i, i+1)$ within layer k; $(\alpha_{(i,k)} = \alpha_k, \forall\ i, k)$

$Q_{(i,k),(j,k),t}$: lateral flow of groundwater in period t among users (i,k) and (j,k)

$Q_{(i,j),(i,k),t}$: lateral flow of groundwater in period t among users (i,j) and (i,k)

a_t: output price of the crop when the crop production quantity is zero in period t for all users

b_t: rate of decrease in crop's output price with respect to the crop's production in period t for all users

c_t: unit cost of groundwater extraction (pumpage) in period t for all users

$g_{(i,k),t}(u_{(i,k),t}, x_{(i,k),t})$: groundwater usage profit function for user (i,k) in period t

$\Gamma_{(i,k),t}(\vec{u}_t, \vec{x}_t)$: total discounted profit from groundwater usage for user (i,k) in period t

$\Gamma^*_{(i,k),t}(\vec{u}_t, \vec{x}_t)$: maximum total discounted profit from groundwater usage for user (i,k) in period t

Grid Configuration

$x_{(i,j,k),t}$: groundwater stock level at the beginning of period t for user (i,j,k) on the grid

$x_{(i,j,k),0}$: initial groundwater stock level at the beginning of the planning horizon for user (i,j,k) on the grid; $(x_{(i,j,k),0} = 1, \forall\ i,j,k)$

$u_{(i,j,k),t}$: groundwater pumpage (and usage) quantity by user (i,j,k) on the grid in period t

$u^*_{(i,j,k),t}$: groundwater optimal pumpage (and usage) quantity by user (i,j,k) on the grid in period t

Salvage Value Function

$sv_{i,2}(u_{i,2}, x_{i,2})$: salvage value function for user i in period two

$\pi_{i,1}, \pi_{i,2}$: respectively, liner and quadratic coefficients of $sv_{i,2}(u_{i,2}, x_{i,2})$

$\tilde{g}_{i,2}(u_{i,2}, x_{i,2})$: sum of the profit realized from groundwater usage in period two and from the salvage value; ($\tilde{g}_{i,2}(u_{i,2}, x_{i,2}) = g_{i,2}(u_{i,2}, x_{i,2}) + sv_{i,2}(u_{i,2}, x_{i,2})$)

User i has access to an underground water stock of $x_{i,t}$ at the beginning of period t, $i = 1, \cdots, n$ and $t = 1, \cdots, T$. There is also an aquifer recharge $w_{i,1} = w_1$ for all i at the beginning of period 2; we assume that recharge does not alleviate the underground water level above the base level $x_{i,0}$. We allow the cost and revenue parameters to vary over time among users. Let $u_{i,t}$ denote the amount of groundwater pumped (and used) by user i, $i = 1, \cdots, n$, in period t, $t = 1, \cdots, T$. It is assumed that $u_{i,t} \leq x_{i,t}$, which implies that groundwater is essentially a private resource within each period and a user can not access groundwater lying beneath another user. In our analysis, we take ($T = 2$) unless stated otherwise. That is, like Saak and Peterson [52], we focus on the case of two successive periods in which multiple users make groundwater usage decisions under both centralized and decentralized settings. Saak and Peterson [52] justify the two-period framework by showing, for an infinite-time horizon, the useful life of the groundwater aquifer may increase or decrease when users are better informed about the hydrology of the region depending on the water usage profit function and the discount rate. Therefore, for the sake of exposition of comparing users' usage behavior under the centralized and decentralized management schemes of the groundwater aquifer, we restrict our time horizon to two successive periods. In the context of water usage for agricultural (irrigation) purposes, a period is defined as an irrigation season, where the initial period (first irrigation season) is designated period 1 and the terminal period (terminal irrigation season) is designated period 2.

As water levels change locally due to consumption by each user, water in the aquifer may flow laterally between adjacent users (between the adjacent areas corresponding to the users' plots). The inter-period lateral flow of groundwater between adjacent users is governed by Darcy's Law. This natural law states that the rate of flow of groundwater through a certain medium (soil) is proportionally related to the hydrologic gradient (*i.e.* the driving force acting on water) and the aquifer's lateral flow (transmissivity) coefficient (*i.e.* the measure of the ability of medium to transmit water), α, as stated in Hornberger et al. [31]. Between two different columns of groundwater, lateral flow of groundwater starts from the column of higher head towards that of shorter head, where α equals the hydrologic conductivity of the medium (soil) multiplied by the cross-sectional area of the two columns' heads (cross-sectional area of the hydrologic gradient) and divided by the distance between the centers of the two water columns. The water stock level of a user in a period will be expressed as a function of the previous period's stock level of the user, the groundwater usage of the user and the neighbors, as well as the aquifer's hydrological properties. In the analysis below, we assume that initial water stocks $x_{i,1}$, are identical for all users $i = 1, ..., n$. Furthermore, the soil properties are assumed similar so that all users' water stocks are subject to the same α, which means that the groundwater aquifer is homogenous, isotropic (*i.e.* hydrologic conductivity is the same in all directions) and the groundwater basin has parallel sides with a flat bottom. Information about the lateral transmissivity coefficient, α, of the common aquifer is assumed to be symmetric across users in period 1 (*i.e.* users know with certainty the lateral flow (transmissivity) coefficient α in period 1). The interaction in the availabilities of groundwater stocks among users makes their decentralized and centralized problems non-separable.

In the context of agricultural water usage, it is assumed that the pumped underground water is used for irrigation of crops. The general profit function of agricultural water usage is given by $\nu_{i,t}(\rho_{i,t}y_{i,t}(u_{i,t}) - \tau_{i,t}(u_{i,t}, x_{i,t}) - k_{i,t})$, which has an empirical estimated specification in Peterson and Ding [48], where $\nu_{i,t}$ is utility-of-income function, $\rho_{i,t}$ is the price per unit of the crop, $y_{i,t}$ is the yield of the crop which is dependent on the amount of water used, $\tau_{i,t}(u_{i,t}, x_{i,t})$ is the cost of pumped groundwater (a joint function of water usage and groundwater stock level) and $k_{i,t}$ is the fixed cost of infra-structural (farming) inputs. We assume a linear utility-of-income function, ($\nu_{i,t}(z) = z$), a quadratic yield function with parameters $a_{i,t}$ and $b_{i,t}$ given by $y_{i,t}(u_{i,t}) = (a_{i,t} - 0.5b_{i,t}u_{i,t})u_{i,t}$, where $(a_{i,t} - 0.5b_{i,t}u_{i,t})$ is the output price of one unit of a crop irrigated by groundwater quantity $u_{i,t}$. This price is linearly decreasing with $u_{i,t}$, where the parameter $a_{i,t}$ is the output price of the crop when the crop production quantity is zero

($i.e.$ when $u_{i,t} = 0$) and parameter $b_{i,t}$ represents the rate of decrease in crop's output price with respect to the crop's production ($i.e.$ when $u_{i,t}$ increases). Hence, the periodic revenue (yield) that could be achieved from irrigating crops is the crop's unit price multiplied by the groundwater quantity, $u_{i,t}$, pumped (and used) in irrigation in period t. Also, we assume a quadratic groundwater extraction cost, with unit extraction (pumpage) cost $c_{i,t}$, given by $\tau_{i,t}(u_{i,t}, x_{i,t}) = \int_0^{u_{i,t}} (x_{i,0} - x_{i,t} + z) dz = c_{i,t}[(x_{i,0} - x_{i,t}) u_{i,t} + 0.5 u_{i,t}^2]$, which increases with the initial depth from the land surface to the water table, $(x_{i,0} - x_{i,t})$, and the quantity of water pumped, $u_{i,t}$. We omit the fixed costs ($k_{i,t} = 0$). In the sequel, similar to Saak and Peterson [52], the profit function of groundwater usage realized by user i for time period t is given by

$$g_{i,t}(u_{i,t}, x_{i,t}) = [\rho_{i,t} a_{i,t} - c_{i,t}(x_{i,0} - x_{i,t})] u_{i,t} - 0.5(\rho_{i,t} b_{i,t} + c_{i,t}) u_{i,t}^2 \tag{3.1}$$

where the cost-revenue parameters $\rho_{i,t}, a_{i,t}, b_{i,t}, c_{i,t} > 0$ and satisfy the following condition

$$(\rho_{i,t} b_{i,t} + c_{i,t}) x_{i,0} < \rho_{i,t} a_{i,t} < (2\rho_{i,t} b_{i,t} + c_{i,t}) x_{i,0} \tag{3.2}$$

The condition in Eqn (3.2) on the parameters follows from the models in Saak and Peterson [52] and is needed for some of our structural results herein as for theirs. Eqn (3.2) can be rewritten as $\rho_{i,t} b_{i,t} x_{i,0} < (\rho_{i,t} a_{i,t} - c_{i,t} x_{i,0}) < 2\rho_{i,t} b_{i,t} x_{i,0}$, which gives the upper and lower bounds on the marginal profit for producing one additional unit of a crop, $(\rho_{i,t} a_{i,t} - c_{i,t} x_{i,0})$, when the entire initial stock of water is exhausted. The lower and upper bounds represent, respectively, the marginal revenue (yield) and its double both realized from producing one additional unit of a crop when the entire initial stock of water is pumped. Notice that The condition in Eqn (3.2) is given in terms of the periodic cost-revenue parameters as well as in terms of the initial stock level of groundwater; $x_{i,0}$, which are known in advance.

For this profit expression in Eqn (3.1), we have the following key property.

Lemma 3.1 (Positivity, Continuity, Concavity)

(i) For $u_{i,t} \leq x_{i,t} \leq x_{i,0}$*, the function* $g_{i,t}(u_{i,t}, x_{i,t})$ *is strictly increasing in* $u_{i,t}$*,* $i = 1, \cdots, n,\ t = 1, 2.$

(ii) The function $g_{i,t}(u_{i,t}, x_{i,t})$ *is continuous and concave in* $u_{i,t}$*,* $i = 1, \cdots, n,\ t = 1, 2.$

Proof *See Appendix.*

We construct our models with non-identical users in the general case. The differences among users may be due to differences in the yield and cost parameters of the users. The differences in the yield parameters ($\rho_{i,t}$, $a_{i,t}$ and $b_{i,t}$) among users represent different cropping and irrigation patterns adopted by users, whereas the difference in the cost parameters ($c_{i,t}$ and $k_{i,t}$) represents different technologies and machinery utilized in pumpage groundwater from the common aquifer and in irrigating the grown crops. The geography of the aquifer region and the soil properties (hydrology) of the land being planted and irrigated characterize possible different transmission structures for the users configured over the common aquifer. Additionally, the specific configuration of the users over this aquifer contribute to the water dynamics over time among users. In this work, we consider two configurations - the strip and ring configurations - within the general framework as outlined above.

3.2 Strip Configuration

We consider the system of n non-identical users distributed adjacently in a strip over the common groundwater aquifer. The setting may be envisioned as an abstraction of a more complex geographic configuration with the only restriction that each user has at most two neighbors. Figure 3.1 depicts the hydrology of the aquifer in the strip configuration.

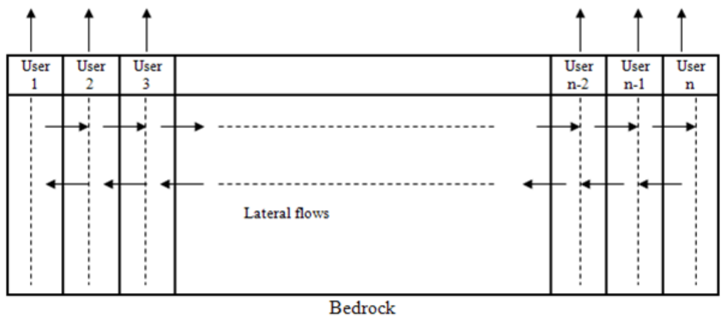
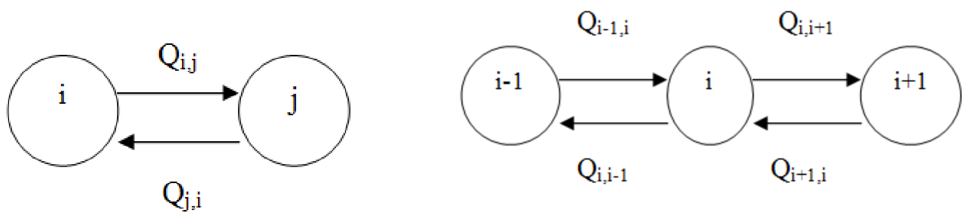


Figure 3.1: Hydrology of the aquifer in the strip configuration

For one dimensional flow of groundwater, there will be lateral flow of groundwater among adjacently located users. Then, the extreme users on the strip (the first and the last) will receive groundwater flow only from one neighbor, whereas for all other (non-extreme) users, flow will be from the two neighbors on both sides. Hence, for $i = 1$ and $j = 2$ and, $i = n$ and $j = n - 1$, as depicted in Figure 3.2a, the lateral flow of groundwater in period 1, $Q_{j,i}$, is given by $Q_{j,i} = -\alpha[(x_{i,1} - u_{i,1} + w_{i,1}) - (x_{j,1} - u_{j,1} + w_{j,1})] = \alpha(u_{i,1} - u_{j,1})$, where, from Saak and Peterson [52], $\alpha \in [0, 0.5]$ is the lateral flow (aquifer transmissivity) coefficient, summarizing the hydrologic dynamics of the groundwater aquifer, and $(x_{i,1} - u_{i,1} + w_{i,1}) - (x_{j,1} - u_{j,1} + w_{j,1})$ is the hydrologic gradient (the difference in hydrologic head between the wells). The minimum value of α corresponds to the purely private resource of groundwater, while the maximum value corresponds to the inter-seasonally common property resource of groundwater (*i.e.* infinite transmissivity). Similarly, by applying Darcy's Law in period 1, a non-extreme user i, $i = 1, \cdots, n - 1$, would have lateral inflows $Q_{i-1,i}$ and $Q_{i+1,i}$, where $Q_{i-1,i} = \alpha(u_{i,1} - u_{i-1,1})$ and $Q_{i+1,i} = \alpha(u_{i,1} - u_{i+1,1})$, as shown in Figure 3.2b.

In this configuration, we consider below the two kinds of decision making - decentralized and centralized problems.



(a) Extreme users

(b) Non-extreme users

Figure 3.2: Lateral flow of groundwater among users in the strip configuration

3.2.1 The Decentralized Problem

In the decentralized problem, each user has the objective of maximizing his/her own total discounted profit over the horizon of two periods by choosing the water usage quantity in each period. But, at the same time, each user has to take into account usages of all other users due to the commonality of the underground aquifer. This generates an $n-$player normal-form game, where the water usage quantity in each period is the strategy of a player (a user), and the payoff function is given by a user's expected total discounted profit over the horizon. The strategy space of any user is constructed from the other users' decisions of water usage and the available (and finite) underground water stocks in any period. In this section, we consider this game-theoretic model and investigate its properties. The decentralized problem above can be stated as a dynamic program as follows. Let $\Gamma^*_{i,t}(\vec{u}_t, \vec{x}_t)$ denote the maximum expected total profit under an optimal water usage schedule for user i for periods t through the end of horizon, where $\vec{u}_t = (u_{1,t}, \cdots, u_{n,t})^T$ is the water usage vector for all users in period t and $\vec{x}_t = (x_{1,t}, \cdots, x_{n,t})^T$ is the water stock vector for all users at the beginning of period t. For $t = 1, 2$, the decentralized problem of user i, $i = 1, \cdots, n$, is solved by the following dynamic program

$$\Gamma^*_{i,t}(\vec{u}_t, \vec{x}_t) = \max_{u_{i,t}} \Gamma_{i,t}(\vec{u}_t, \vec{x}_t) = \max_{u_{i,t}} [g_{i,t}(u_{i,t}, x_{i,t}) + \beta_{i,t}\Gamma^*_{i,t+1}(\vec{u}_{t+1}, \vec{x}_{t+1})] \quad (3.3)$$

s.t.

$$x_{i,t+1} = \begin{cases} x_{i,t} + w_{i,t} - (1-\alpha)u_{i,t} - \alpha u_{j,t}, & (i,j) \in \{(1,2),(n,n-1)\} \\ x_{i,t} + w_{i,t} - (1-2\alpha)u_{i,t} - \alpha(u_{i-1,t} + u_{i+1,t}), & i = 2,\cdots,n-1 \end{cases} \tag{3.4}$$

$$0 \le u_{i,t} \le x_{i,t} \tag{3.5}$$

In the above problem, the decision variables, $u_{i,t}$, are the water usage quantities of user i in period t, $i = 1,\cdots,n$, $t = 1,2$. Eqn (3.4) corresponds to the recursive temporal relationship among the water stocks of the users as dictated by Darcy's Law. More specifically, the water stock level of user i at the beginning of period $t+1$; $x_{i,t+1}$, equals to that at the beginning of period t; $x_{i,t}$, plus the aquifer recharge; $w_{i,t}$, plus the lateral flows of groundwater from adjacent users to user i; Q (dedicated by Darcy's Law), minus the water usage (pumpage) quantity $u_{i,t}$ in period t, for $i = 1,\cdots,n$ and $t = 1,2$. The first part of Eqn (3.4) gives the water stock balance equation for the extreme users while the second one gives that for the non-extreme users in the strip. Eqn (3.5) gives the constraint for each user's water usage, that is, in any period no user in the system can pump more than her available water stock at the beginning of that period. For $i = 1,\cdots,n$, $t = 1,2$, we assume that the discount rate $\beta_{i,t} = \beta$ with $0 \le \beta \le 1$, $x_{i,1} = x_1$, $w_{i,1} = w_1$ and $\Gamma^*_{i,3}(\vec{u}_3,\vec{x}_3) \equiv 0$ for all $\vec{x}_3$, $\vec{u}_3$. We later relax the condition on $\Gamma^*_3(.,.)$ in Section 3.7.

We are now ready to examine some properties of the optimal solution to the above formulation. We first provide the structural results for the objective function, $\Gamma_{i,t}(\vec{u}_t,\vec{x}_t)$. From Lemma 3.1 (i), immediately we have the following.

Corollary 3.1 *The within-period profit function $g_{i,t}(u_{i,t},x_{i,t})$ attains its maximum at $u^*_{i,t} = x_{i,t}$, $i = 1,\cdots,n$, $t = 1,2$.*

This result has two implications. *(i)* The myopic solution of the problem is trivial; that is, all water resources are depleted in the first period for any length of the horizon. *(ii)* In the optimal solution, all users deplete their water resources in the very last period, (*i.e.*, $u^*_{i,2} = x_{i,2}$, $\forall i$). Therefore, we have

$\Gamma_{i,1}(\vec{u}_1, \vec{x}_1) = [g_{i,1}(u_{i,1}, x_{i,1}) + \beta g_{i,2}(x_{i,2}, x_{i,2})]$. Furthermore, $x_{i,2}$ is a function of $\vec{u}_1$; and, hence, the $n-$user problem given in (3.3)-(3.5) reduces to a single period problem which is only a function of $\vec{u}_1$ and $\vec{x}_1$. We can use these implications to obtain below a tighter formulation of the original problem and to establish additional properties.

Proposition 3.1 (Positivity, Continuity, Concavity)

(i) $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$ *is strictly increasing in* $u_{i,1}$ *at* $u_{i,1} = 0$ *if* $\rho_{i,1}a_{i,1} \geq \beta(\rho_{i,2}a_{i,2} + c_{i,2}w_1)$*,* $i = 1, \cdots, n$*.*

(ii) $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$ *is continuous and jointly concave in* $\vec{u}_1$ *if and only if* $c_{i,2} \leq \rho_{i,2}b_{i,2}$*,* $i = 1, \cdots, n$*.*

Proof *See Appendix.*

The first part of the above result establishes the positivity of the optimal solution, that is $u^*_{i,1} > 0$ for all i, $i = 1, \cdots, n$. Therefore, it suffices for our setting to consider a tighter search space ($0 < u_{i,1} \leq x_1\ \forall i$). Positivity of $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$ is guaranteed when the output price of the irrigated crop in the initial irrigation season (period 1) is at least greater than the discounted output price and the pumping cost in the terminal irrigation season (period 2). This is intuitive because at the beginning of the season, crop's output (yield) is lower than its output during the second season, the reason that makes its price greater at the beginning of crop production as, in the first season, the supply is smaller than the market demand. The latter part guarantees a well-behaving objective function for optimization. Concavity of $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$ is guaranteed when the concavity of $g_{i,2}(x_{i,2}, x_{i,2})$ exists. Now, as stated in the latter part of the above result, the concavity of $g_{i,2}(x_{i,2}, x_{i,2})$ is ensured when the marginal pumping cost of groundwater in the second period is lower than the marginal revenue (crop yield) in the same period. We can now re-state the two-period decentralized problem as follows. For $i = 1, \cdots, n$,

$$\max_{u_{i,1}} \Gamma_{i,1}(\vec{u}_1, \vec{x}_1) = \max_{u_{i,1}} [g_{i,1}(u_{i,1}, x_{i,1}) + \beta g_{i,2}(x_{i,2}, x_{i,2})] \tag{3.6}$$

$$s.t. \quad 0 < u_{i,1} \leq x_1 \tag{3.7}$$

where the water stock in the last period $x_{i,2}$ is given by Eqn (3.4).

We note that the problem stated in Eqns (3.4), (3.6) and (3.7) corresponds to a single period strategic form game given by the payoff function $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$ and the strategy set $u_{i,1}$. We observe that the strategy set; $0 < u_{i,1} \leq x_1$, is nonempty, continuous, convex and compact (closed and bounded) and that the payoff function is continuous and jointly concave in the players' strategies as implied by Proposition 3.1. Then, from Theorem 1 in Dasgubta and Maskin [16], we have the following result.

Proposition 3.2 (Existence of Nash Equilibrium) *The $n-$player game which corresponds to the decentralized problem in the strip configuration has (at least one) Nash equilibrium.*

A Nash equilibrium corresponds to the simultaneous solution of n constrained optimization problems given above. If the Nash equilibrium occurs such that no user depletes his initial water stock in the first period ($u^*_{i,1} < x_1$, $\forall i$), then we have the *unconstrained* solution. Although it cannot be guaranteed in general, this result appears to us as the most common, real-life solution. We are able to obtain further structural results and elegant solutions for the unconstrained optimization problem, which we shall present shortly. For completeness, we need also to consider the case of *constrained* solutions where $u^*_{i,1} = x_1$. To this end, we construct the Lagrange function $L(u_{i,1}, \delta_i) = \Gamma_{i,1}(\vec{u}_1, \vec{x}_1) + \delta_i(x_1 - u_{i,1})$, where $\delta_i \geq 0$ is the Lagrange multiplier corresponding to the constraint $u_{i,1} \leq x_1$. Let $\vec{u}^*_1 = (u^*_{1,1}, \cdots, u^*_{n,1})^T$ be the vector of optimal water usage in period 1, $\vec{\delta}^* = (\delta^*_1, \cdots, \delta^*_n)^T$ be the optimal vector of the Lagrange multipliers, $\vec{x}_1 = (x_1, \cdots, x_1)^T$ be an $n \times 1$ vector of initial water stock in period 1 and $\vec{0} = (0, \cdots, 0)^T$ be an $n \times 1$ zero vector. Then, as shown in the Appendix for Proposition 3.3, the Karush-Kuhn-Tucker (KKT) conditions of the Lagrange function give the following.

$$A\vec{u}^*_1 - \vec{\delta}^{*T} = W \tag{3.8}$$

$$\vec{\delta}^{*T}(\vec{x}_1 - \vec{u}_1^*) = 0 \tag{3.9}$$

$$\vec{u}_1^* \leq \vec{x}_1 \tag{3.10}$$

$$\vec{\delta}^* \geq \vec{0} \tag{3.11}$$

where $A_{n\times n} = \begin{pmatrix} \gamma_1 & \sigma_1 & 0 & 0 & \dots & 0 \\ \epsilon_1 & \gamma_2 & \sigma_2 & 0 & \dots & 0 \\ 0 & \epsilon_2 & \gamma_3 & \sigma_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \epsilon_{n-2} & \gamma_{n-1} & \sigma_{n-1} \\ 0 & 0 & \dots & 0 & \epsilon_{n-1} & \gamma_n \end{pmatrix}$,

$W_{n\times 1} = \begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_{n-1} & \lambda_n \end{pmatrix}^T$,

$$\gamma_i = \begin{cases} \beta(1-\alpha)^2(c_{i,2} - \rho_{i,2}b_{i,2}) - (\rho_{i,1}b_{i,1} + c_{i,1}), & i = 1, n \\ \beta(1-2\alpha)^2(c_{i,2} - \rho_{i,2}b_{i,2}) - (\rho_{i,1}b_{i,1} + c_{i,1}), & \text{o.w.} \end{cases},$$

$$\lambda_i = \begin{cases} \beta(1-\alpha)[\rho_{i,2}(a_{i,2} - b_{i,2}x_1) + (c_{i,2} - \rho_{i,2}b_{i,2})w_1] - \rho_{i,1}a_{i,1}, & i = 1, n \\ \beta(1-2\alpha)[\rho_{i,2}(a_{i,2} - b_{i,2}x_1) + (c_{i,2} - \rho_{i,2}b_{i,2})w_1] - \rho_{i,1}a_{i,1}, & \text{o.w.} \end{cases},$$

$$\sigma_i = \begin{cases} \beta\alpha(1-\alpha)(c_{i,2} - \rho_{i,2}b_{i,2}), & i = 1 \\ \beta\alpha(1-2\alpha)(c_{i,2} - \rho_{i,2}b_{i,2}), & \text{o.w.} \end{cases} \quad \text{and}$$

$$\epsilon_i = \begin{cases} \beta\alpha(1-\alpha)(c_{n,2} - \rho_{n,2}b_{n,2}), & i = n-1 \\ \beta\alpha(1-2\alpha)(c_{i,2} - \rho_{i,2}b_{i,2}), & \text{o.w.} \end{cases}$$

Proposition 3.1 implies that the Hessian of $\Gamma_{i,1}$ is negative semi-definite, and, hence, the two-period decentralized problem is a concave quadratic program. Therefore, the KKT conditions in (3.8)-(3.11) are, in fact, sufficient for $\vec{u}_1^*$ to be a global optimal solution as mentioned in Nocedal and Wright [46]. Several classes of algorithms have been used for solving concave quadratic problems that contain both inequality and equality constraints. Active-set methods have for long been used and are proved to be effective for small- and medium-sized problems. However, a special type of active-set methods called the gradient projection method has recently been shown most effective for solving concave quadratic problems having only upper and lower bounds as constraints on the decision variables, as discussed in Nocedal and Wright [46]. Hence, any one of these methods may be employed for solving the KKT conditions above since we have only the upper bound on decision variables. Clearly, if $\delta_i^* = 0$ in the solution for the above Lagrange function for all i, then, the optimal solution is the unconstrained solution (global maximizer of $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$ in $\Re^+$), which we consider next. First, we establish the uniqueness of the unconstrained optimal solution.

Proposition 3.3 (Uniqueness of the global maximizer and optimality)

(i) The global maximizer of $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$ *is unique and given by* $u_{1,1}^{**} = \frac{\tau_1}{\kappa}$, $u_{2,1}^{**} = \frac{\tau_2}{\kappa}$ *and* $u_{k+2,1}^{**} = \hat{\lambda}_{k+2} + \hat{e}_{(k+2,1)} u_{1,1}^{**} + \hat{e}_{(k+2,2)} u_{2,1}^{**}$, *for* $k = 1, \cdots, n-2$, *where* $\tau_1 = \lambda_1 [\sum_{j=0}^{1} e_{(n,n-j)} \hat{e}_{(n-j,2)}] - \sigma_1 [\lambda_n - \sum_{j=0}^{1} e_{(n,n-j)} \hat{\lambda}_{n-j}]$,
$\tau_2 = \gamma_1 [\lambda_n - \sum_{j=0}^{1} e_{(n,n-j)} \hat{\lambda}_{n-j}] - \lambda_1 [\sum_{j=0}^{1} e_{(n,n-j)} \hat{e}_{(n-j,1)}]$,
$\kappa = \gamma_1 [\sum_{j=0}^{1} e_{(n,n-j)} \hat{e}_{(n-j,2)}] - \sigma_1 [\sum_{j=0}^{1} e_{(n,n-j)} \hat{e}_{(n-j,1)}]$,
$\hat{\lambda}_{k+2} = [\lambda_{k+1} - \sum_{j=1}^{2} e_{(k+1,k+2-j)} \hat{\lambda}_{k+2-j}]/[e_{(k+1,k+2)}]$,
$\hat{e}_{(k+2,m)} = -[\sum_{j=1}^{2} e_{(k+1,k+2-j)} \hat{e}_{(k+2-j,m)}]/[e_{(k+1,k+2)}]$, *for* $m = 1, 2$; $\hat{\lambda}_1 = \hat{\lambda}_2 = 0$, $\hat{e}_{(1,1)} = \hat{e}_{(2,2)} = 1$, $\hat{e}_{(1,2)} = \hat{e}_{(2,1)} = 0$ *and* $e_{(m,i)} = \hat{e}_{(m,1)} = \hat{e}_{(m,2)} = 0$, *for* $\{i, m\} < 1$ *and* $\{i, m\} > n$; *for* $i = 1, \cdots, n$, $e_{(i,i)} = \gamma_i$ *and*

$$e_{(i,j)} = \begin{cases} \sigma_i, & (i,j) = (i, i+1),\ i = 1, \cdots, n-1 \\ \epsilon_i, & (i,j) = (i, i-1),\ i = 2, \cdots, n \\ 0, & o.w. \end{cases} .$$

(ii) If $0 \le u_{i,1}^{**} \le x_1$, *for all* i, *then* $u_{i,1}^{**}$, *given above, is the optimal solution for the decentralized problem.*

Proof *See Appendix.*

When all users are identical, we have

$$\gamma_i = \begin{cases} \gamma, & i = 1, n \\ \epsilon, & \text{o.w.} \end{cases}, \quad \sigma_i = \begin{cases} \omega, & i = 1 \\ \sigma, & \text{o.w.} \end{cases}, \quad \epsilon_i = \begin{cases} \omega, & i = n-1 \\ \sigma, & \text{o.w.} \end{cases} \quad \text{and}$$

$$\lambda_i = \begin{cases} \eta, & i = 1, n \\ \lambda, & \text{o.w.} \end{cases}, \text{ where } \gamma = \beta(1-\alpha)^2(c_2 - \rho_2 b_2) - (\rho_1 b_1 + c_1),$$

$\epsilon = \beta(1-2\alpha)^2(c_2 - \rho_2 b_2) - (\rho_1 b_1 + c_1)$, $\omega = \beta\alpha(1-\alpha)(c_2 - \rho_2 b_2)$,

$\sigma = \beta\alpha(1-2\alpha)(c_2 - \rho_2 b_2)$, $\eta = \beta(1-\alpha)[\rho_2(a_2 - b_2 x_1) + (c_2 + \rho_2 b_2)w_1] - \rho_1 a_1$

and $\lambda = \beta(1-2\alpha)[\rho_2(a_2 - b_2 x_1) + (c_2 - \rho_2 b_2)w_1] - \rho_1 a_1$.

In this case, we have a closed form result for the optimal solution to the unconstrained problem.

Corollary 3.2 (Unique global maximizer for identical users) *For $n-$identical users on a strip, let $k = n/2$ if n is even and $(n+1)/2$ otherwise. Then, the system $A\vec{u}_1^{**} = W$ has a unique solution given by $u_{i,1}^{**} = u_{n-i+1,1}^{**} = h_0 + h_1(r_1)^i + h_2(r_2)^i$, $i = 1, \cdots, k$,*

where $h_0 = \lambda/(2\sigma+\epsilon)$, $r_1 = (-\epsilon - \sqrt{\epsilon^2 - 4\sigma^2})/2\sigma$, $r_2 = (-\epsilon + \sqrt{\epsilon^2 - 4\sigma^2})/2\sigma$ and

for $k = n/2$,

$$h_1 = \frac{[\eta - (\frac{\gamma+\omega}{2\sigma+\epsilon})\lambda]}{[\gamma r_1 + \omega(r_1)^2] - [\gamma r_2 + \omega(r_2)^2][\frac{\sigma+(\sigma+\epsilon)r_1}{\sigma+(\sigma+\epsilon)r_2}](\frac{r_1}{r_2})^{(k-1)}};\ h_2 = -h_1[\frac{\sigma+(\sigma+\epsilon)r_1}{\sigma+(\sigma+\epsilon)r_2}](\frac{r_1}{r_2})^{(k-1)}$$

and for $k = (n+1)/2$,

$$h_1 = \frac{[\eta - (\frac{\gamma+\omega}{2\sigma+\epsilon})\lambda]}{[\gamma r_1 + \omega(r_1)^2] - [\gamma r_2 + \omega(r_2)^2][\frac{2\sigma+\epsilon r_1}{2\sigma+\epsilon r_2}](\frac{r_1}{r_2})^{(k-1)}};\ h_2 = -h_1[\frac{2\sigma+\epsilon r_1}{2\sigma+\epsilon r_2}](\frac{r_1}{r_2})^{(k-1)}.$$

Proof *See Appendix.*

We note that Saak and Peterson [52] find the Nash equilibrium for $n = 2$, which gives water usages for both users that are symmetric, unique and dependent on lateral flow coefficient α. Corollary 3.2 also implies that the unconstrained optimal solution is symmetric around the mid-point(s) of the strip and generalizes their findings to the case where $n > 2$. Besides, since ϵ and σ are negative, we have $r_1, r_2 < 0$ and $r_1 > r_2$. This implies that the unconstrained optimal solution has a fluctuating structure across the users from the extremes toward the center. Thus, for the unconstrained optimal solution, we have established theoretically Saak and Peterson's [52] conjecture (p. 226) that water usage would not be monotone for multiple users ($n > 2$) even when they are all identical. We think that this has significance for policy makers in the design of payment schemes (cost structures) for underground water usage for multiple users ($n > 2$). In our numerical study that will be presented in the next chapter, we have observed that, typically, the second most extreme users at both ends of the strip have the highest water consumption in the unconstrained solutions. If this observation always holds, then it may be possible to obtain the cost-revenue parameter space so that the Nash equilibrium always occurs as the unconstrained optimal.

In the above formulation of the decentralized problem, we have assumed that users have complete (perfect) information about other players' parameters and the hydrological properties of the aquifer expressed through α. An interesting variant of the problem analyzed by Saak and Peterson [52] for $n = 2$ is the case where users have *incomplete* information about α considered to be a random variable. In the case of identical users, it turns out that, also for $n > 2$, the problem can be stated as the expected total discounted profits and all of the results provided so far involving α would still hold in the expectation sense; that is, $E[\alpha]$ in place of α, $E[(1-\alpha)^2]$ in place of $(1-\alpha)^2$ *etc.* For non-identical users, incorporation of asymmetry of information seems not so straightforward. We examine further properties of the optimal solutions in our numerical study in the next chapter.

3.2.2 The Centralized Problem

In this problem, we envision a central decision maker (social planner in the public policy parlance) aiming at determining the optimal water usage for each user so that the total joint discounted profit of all users throughout the planning horizon of two periods is maximized. The problem can be stated as a dynamic programming problem as follows. For $t = 1, 2$,

$$\tilde{\Gamma}^*_t(\vec{u}_t,\vec{x}_t) = \max_{u_{1,t},\cdots,u_{n,t}} \tilde{\Gamma}_t(\vec{u}_t,\vec{x}_t) = \max_{u_{1,t},\cdots,u_{n,t}} \{\{\sum_{i=1}^{n} g_{i,t}(u_{i,t},x_{i,t})\} + \beta_t \tilde{\Gamma}^*_{t+1}(\vec{u}_{t+1},\vec{x}_{t+1})\} \tag{3.12}$$

$$s.t. \;\; (3.4) \;\; and \;\; (3.7)$$

where $\tilde{\Gamma}_t(\vec{u}_t,\vec{x}_t)$ is the joint profit-to-go function from period t until the end of the problem horizon. All of the other conventions and notations of the decentralized problem are retained. Since $\tilde{\Gamma}_t(\vec{u}_t,\vec{x}_t)$ is a positive linear combination of individual discounted profit-to-go functions in the decentralized problem, we immediately have the following.

Corollary 3.3 (Myopic optimality, Positivity, Continuity, Concavity)

(i) The myopically optimal water usage in period t is to deplete all stock $[g^*_{i,t}(u_{i,t},x_{i,t}) = g_{i,t}(x_{i,t},x_{i,t})]$.

(ii) For a given $\vec{x}_1$, $\tilde{\Gamma}_1(\vec{u}_1,\vec{x}_1)$ is strictly increasing in $u_{i,1}$ at $u_{i,1}=0$ if $\rho_{i,1}a_{i,1} \geq \beta(\rho_{i,2}a_{i,2}+c_{i,2}w_1)$, for all i.

(iii) For a given $\vec{x}_1$, $\tilde{\Gamma}_1(\vec{u}_1,\vec{x}_1)$ is continuous and jointly concave in $\vec{u}_1$ if and only if $c_{i,2} \leq \rho_{i,2}b_{i,2}$, for all i.

The above result implies that the centralized problem also reduces to an equivalent single period concave quadratic optimization problem subject to the constraint set $0 < u_{i,1} \leq x_1$, for all i. Constructing the Lagrange function for this problem $L(u_{i,1},\delta_i) = \tilde{\Gamma}_i(\vec{u}_t,\vec{x}_t) + \delta_i(x_1 - u_{i,1})$, the KKT conditions result in $\tilde{A}\vec{u}^*_1 - \vec{\delta}^{*T} = \tilde{W}$, together with (3.9)-(3.11). The unconstrained solution of the centralized problem corresponding to the general case of non-identical users is given in the following result.

Proposition 3.4 (Uniqueness of the global maximizer and optimality)

(i) The global maximizer of $\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1)$ *is unique and given by* $u_{1,1}^{**} = \frac{\tilde{\lambda}_1}{\tilde{\omega}}$, $u_{2,1}^{**} = \frac{\tilde{\lambda}_2}{\tilde{\omega}}$ *and* $u_{k+2,1}^{**} = \hat{\theta}_{k+2} + \hat{e}_{(k+2,1)} u_{1,1}^{**} + \hat{e}_{(k+2,2)} u_{2,1}^{**}$, *for* $k = 1, \cdots, n-2$, *where*

$\tilde{\lambda}_1 = \sum_{j=0}^{2} e_{(n,n-j)} \hat{e}_{(n-j,2)} [\theta_{n-1} - \sum_{j=0}^{3} e_{(n-1,n-j)} \hat{\theta}_{n-j}] - \sum_{j=0}^{3} e_{(n-1,n-j)} \hat{e}_{(n-j,2)} [\theta_n - \sum_{j=0}^{2} e_{(n,n-j)} \hat{\theta}_{n-j}]$,

$\tilde{\lambda}_2 = \sum_{j=0}^{3} e_{(n-1,n-j)} \hat{e}_{(n-j,1)} [\theta_n - \sum_{j=0}^{2} e_{(n,n-j)} \hat{\theta}_{n-j}] - \sum_{j=0}^{2} e_{(n,n-j)} \hat{e}_{(n-j,1)} [\theta_{n-1} - \sum_{j=0}^{3} e_{(n-1,n-j)} \hat{\theta}_{n-j}]$,

$\tilde{\omega} = \sum_{j=0}^{3} e_{(n-1,n-j)} \hat{e}_{(n-j,1)} \sum_{j=0}^{2} e_{(n,n-j)} \hat{e}_{(n-j,2)}$
$- \sum_{j=0}^{3} e_{(n-1,n-j)} \hat{e}_{(n-j,2)} \sum_{j=0}^{2} e_{(n,n-j)} \hat{e}_{(n-j,1)}$,

$\hat{\theta}_{k+2} = [\theta_k - \sum_{j=1}^{4} e_{(k,k+2-j)} \hat{\theta}_{k+2-j}] / [e_{(k,k+2)}]$,

$\hat{e}_{(k+2,m)} = -[\sum_{j=1}^{4} e_{(k,k+2-j)} \hat{e}_{(k+2-j,m)}] / [e_{(k,k+2)}]$, *for* $m = 1, 2$; $\hat{\theta}_1 = \hat{\theta}_2 = 0$, $\hat{e}_{(1,1)} = \hat{e}_{(2,2)} = 1$, $\hat{e}_{(1,2)} = \hat{e}_{(2,1)} = 0$ *and* $e_{(m,i)} = \hat{e}_{(m,1)} = \hat{e}_{(m,2)} = 0$, *for* $\{m, i\} < 1$ *and* $\{m, i\} > n$; $e_{(i,i)} = (\rho_{i,1} b_{i,1} + c_{i,1}) + \beta(1-\alpha)^2 (\rho_{i,2} b_{i,2} - c_{i,2}) + \beta\alpha^2 (\rho_{j,2} b_{j,2} - c_{j,2})$, $(i, j) \in \{(1, 2), (n, n-1)\}$,

$e_{(i,i)} = (\rho_{i,1} b_{i,1} + c_{i,1}) + \beta(1-2\alpha)^2 (\rho_{i,2} b_{i,2} - c_{i,2}) + \beta\alpha^2 [(\rho_{i-1,2} b_{i-1,2} - c_{i-1,2}) + (\rho_{i+1,2} b_{i+1,2} - c_{i+1,2})]$, $i = 2, \cdots, n-1$,

$e_{(i,m)} = \beta\alpha^2 (\rho_{j,2} b_{j,2} - c_{j,2})$, $(i, j, m) \in \{(1, 2, 3), (n, n-1, n-2), (2, 3, 4), (n-1, n-2, n-3), (k, k-1, k-2), (k, k+1, k+2)\}$,

$e_{(i,j)} = \beta\alpha(1-\alpha)(\rho_{i,2} b_{i,2} - c_{i,2}) + \beta\alpha(1-2\alpha)(\rho_{j,2} b_{j,2} - c_{j,2})$, $(i, j) \in \{(1, 2), (n, n-1)\}$,

$e_{(i,j)} = \beta\alpha(1-2\alpha)(\rho_{i,2} b_{i,2} - c_{i,2}) + \beta\alpha(1-\alpha)(\rho_{j,2} b_{j,2} - c_{j,2})$, $(i, j) \in \{(2, 1), (n-1, n)\}$,

$e_{(i,j)} = \beta\alpha(1-2\alpha)[(\rho_{i,2} b_{i,2} - c_{i,2}) + (\rho_{j,2} b_{j,2} - c_{j,2})]$, $(i, j) \in \{(2, 3), (n-1, n-2), (k, k-1), (k, k+1)\}$,

$e_{(i,j)} = 0$, *elsewhere;*

$\theta_i = \rho_{i,1} a_{i,1} - \beta(1-\alpha)(\rho_{i,2} a_{i,2} + c_{i,2} w_1) - \beta\alpha(\rho_{j,2} a_{j,2} + c_{j,2} w_1) + \beta(1-\alpha)\rho_{i,2} b_{i,2}(x_1 + w_1) + \beta\alpha\rho_{j,2} b_{j,2}(x_1 + w_1)$, $(i, j) \in \{(1, 2), (n, n-1)\}$ *and*

$\theta_i = \rho_{i,1}a_{i,1} - \beta(1-2\alpha)(\rho_{i,2}a_{i,2} + c_{i,2}w_1) - \beta\alpha[(\rho_{i-1,2}a_{i-1,2} + c_{i-1,2}w_1) + (\rho_{i+1,2}a_{i+1,2} + c_{i+1}w_1)] + \beta(1-2\alpha)\rho_{i,2}b_{i,2}(x_1+w_1) + \beta\alpha[\rho_{i-1,2}b_{i-1,2}(x_1 + w_1) + \rho_{i+1,2}b_{i+1,2}(x_1+w_1)]$, $i = 2, \cdots, n-1$.

(ii) If $0 \leq u^{**}_{i,1} \leq x_1$*, for all* i*, then* $u^{**}_{i,1}$*, given above, is the optimal solution for the centralized problem.*

Proof *See Appendix.*

For identical users, we establish that the global maximizer of $\tilde{\Gamma}_1$ in $\Re^+$ is unique, independent of the hydrological properties of the aquifer (α) and it is the *same* for all users unlike the decentralized solution. Furthermore, the unconstrained solution is optimal for the centralized problem for certain cost and revenue parameter values. We state this result below.

Corollary 3.4 (Uniqueness of the global maximizer and optimality for identical users)

(i) Suppose that users are identical and $c_2 \leq \rho_2 b_2$*. Then, the global maximizer of* $\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1)$ *is unique and given by*

$$u^{**}_{i,1} = u^{**} = [\rho_1 a_1 - \beta(\rho_2 a_2 + c_2 w_1) + \beta\rho_2 b_2(x_1+w_1)]/[(\rho_1 b_1 + c_1) + \beta(\rho_2 b_2 - c_2)],\ \forall i$$

(ii) If $0 \leq u^{**}_{i,1} \leq x_1$*, for all* i*, then the optimal solution for the centralized problem is given by* u^{**} *above.*

Proof *See Appendix.*

Saak and Peterson [52] have shown for $n = 2$ that the optimal solution is independent of the characteristics of the aquifer expressed through α. Hence, Corollary 3.4 generalizes this finding. However, Saak and Peterson [52] make an implicit assumption that the Nash equilibrium will be the unconstrained solution throughout their analysis. In our result, we establish the conditions for the optimality of the global maximizer to be within the constraint set. The conditions for the optimal solution above imply that, under the cost-revenue assumptions of Saak and Peterson [52], the centralized problem results in an optimal usage which does not deplete the initial stock when $0.5 \leq \beta \leq 1$ - giving a realistic hurdle rate between 0% and 100% per period. Hence, we think that the above optimal result would be observed in most realistic cases. From a policy maker's perspective, it is important to know if the centralized solution can be achieved in the decentralized game- theoretic setting through a pricing mechanism. Under the stated condition above, the optimal solution dictates the same usage for all users. However, in the decentralized solution for the unconstrained case, we established that water usage fluctuates from the ends toward the midpoint(s) of the strip. As these constitute instances of counter examples, we establish by contradiction the following.

Corollary 3.5 (No coordination) *In a strip configuration with n identical users, for $(\rho_t b_t + c_t)x_0 < \rho_t a_t < (2\rho_t b_t + c_t)x_0$, there does not exist a periodic unit pumpage cost c_t that equates the Nash equilibrium with the centralized optimal solution, for $t = 1, 2$.*

We present further observations about the optimal solution in our numerical section in the next chapter.

3.3 Ring Configuration

In this section, we consider the setting where all n users are connected to each other in a ring or circular configuration, as depicted in Figure 3.3. By definition of a ring, we have $n > 2$. Unlike the strip configuration examined above, there are no locational extremes (ends) and each user has exactly two neighbors. Hence, the lateral flows in the aquifer makes all users communicate with each other; and, one particular user's water consumption affects all users in the system either directly or indirectly. The more even nature of the structure brings a similar evenness to the solution as well, as shall be discussed below. Users are numbered in a clockwise fashion where each user has lateral flow from one preceding and one succeeding adjacent user in the ring. In this configuration, we consider below the decentralized and centralized decision making environments.

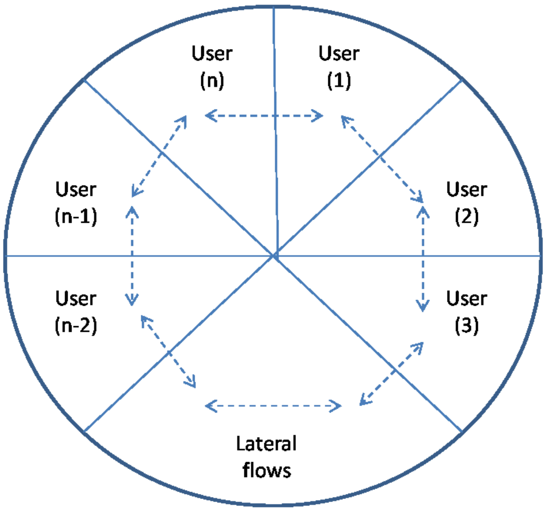


Figure 3.3: Hydrology of the aquifer in the ring configuration

3.3.1 The Decentralized Problem

The decentralized problem for the ring configuration is similar to that for the strip configuration except that the recursive relation between water stocks over time is different owing to the non-existence of any ends of a ring. For $t = 1, 2,$ the decentralized problem for user i, $i = 1, \cdots, n$, is formally stated as a dynamic program given by

$$\Gamma^*_{i,t}(\vec{u}_t, \vec{x}_t) = \max_{u_{i,t}} \Gamma_{i,t}(\vec{u}_t, \vec{x}_t) = \max_{u_{i,t}} [g_{i,t}(u_{i,t}, x_{i,t}) + \beta_{i,t}\Gamma^*_{i,t+1}(\vec{u}_{t+1}, \vec{x}_{t+1})] \quad (3.13)$$

s.t.

$$x_{i,t+1} = \begin{cases} x_{i,t} + w_{i,t} - (1-2\alpha)u_{i,t} - \alpha(u_{j,t} + u_{m,t}), & (i,j,m) = (1,n,2) \\ x_{i,t} + w_{i,t} - (1-2\alpha)u_{i,t} - \alpha(u_{j,t} + u_{m,t}), & (i,j,m) = (n,n-1,1) \\ x_{i,t} + w_{i,t} - (1-2\alpha)u_{i,t} - \alpha(u_{i-1,t} + u_{i+1,t}), & i = 2, \cdots, n-1 \end{cases} \quad (3.14)$$

$$0 \leq u_{i,t} \leq x_{i,t} \tag{3.15}$$

In the above, we retain the previous notations. Note that Eqn (3.14) describes the recursive temporal relationship among the water stocks of the users under Darcy's Law; unlike the strip, the ring configuration allows for each user to communicate with its immediate neighbors. As before, we have the same α for all users and all t; $\beta_{i,t} = \beta$ with $0 \leq \beta \leq 1$; we set $x_{i,1} = x_1$, $w_{i,1} = w_1$ and $\Gamma_3^*(\vec{u}_3, \vec{x}_3) \equiv 0$ for all $\vec{x}_3$, $\vec{u}_3$ and for $i = 1, \cdots, n$. We later relax the condition on $\Gamma_3^*(.,.)$ in Section 3.7.

The properties of the within period profit function in Corollary 3.1 also imply that the decentralized problem in the ring configuration can be written as a single period problem, and that its objective function is also a well-behaving function as stated in the following result.

Proposition 3.5 (Positivity, Continuity, Concavity)

(i) $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$ $(= [g_{i,1}(u_{i,1}, x_{i,1}) + \beta g_{i,2}(x_{i,2}, x_{i,2})])$ *is strictly increasing in* $u_{i,1}$ *at* $u_{i,1} = 0$ *if* $\rho_{i,1}a_{i,1} \geq \beta(\rho_{i,2}a_{i,2} + c_{i,2}w_1)$, $i = 1, \cdots, n$.

(ii) $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$ *is continuous and jointly concave in* $\vec{u}_1$ *if and only if* $c_{i,2} \leq \rho_{i,2}b_{i,2}$, $i = 1, \cdots, n$.

The proof of the first part of the above result is as that Proposition 3.1. Also, the proof of the second part is identical to that in Proposition 3.1 for the non-extreme users. Hence, we omit the proof of the above result. Proposition 3.5 enables a tighter reformulation of the $n-$user problem given by Eqn (3.6) as the objective function subject to Eqn (3.7) where the water stock in the last period $x_{i,2}$ is given by Eqn (3.14). As the properties of the problem satisfy those of Theorem 1 in Dasgubta and Maskin [16], we have the existence of a Nash equilibrium as stated below.

Proposition 3.6 (Existence of Nash Equilibrium) *The* $n-$*player game which corresponds to the decentralized problem in the ring configuration has (at least one) Nash equilibrium.*

The Nash equilibrium corresponds to the simultaneous solution of n constrained optimization problems with a single constraint $u_{i,1} \leq x_1$, $i = 1, \cdots, n$. As shown in the Appendix for Proposition 3.7, the KKT conditions of the Lagrange function $L(u_{i,1}, \delta_i) = \Gamma_i(\vec{u}_1, \vec{x}_1) + \delta_i(x_1 - u_{i,1})$, together with (3.9)-(3.11), give $B\vec{u}_1^* - \vec{\delta}^{*T} = Z$,where

$$B_{n\times n} = \begin{pmatrix} \epsilon_1 & \sigma_1 & 0 & 0 & \dots & \sigma_1 \\ \sigma_2 & \epsilon_2 & \sigma_2 & 0 & \dots & 0 \\ 0 & \sigma_3 & \epsilon_3 & \sigma_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \sigma_{n-1} & \epsilon_{n-1} & \sigma_{n-1} \\ \sigma_n & 0 & \dots & 0 & \sigma_n & \epsilon_n \end{pmatrix}, Z_{n\times 1} = \begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_{n-1} & \lambda_n \end{pmatrix}^T$$

and

$\epsilon_i = \beta(1-2\alpha)^2(c_{i,2} - \rho_{i,2}b_{i,2}) - (\rho_{i,1}b_{i,1} + c_{i,1})$, $\sigma_i = \beta\alpha(1-2\alpha)(c_{i,2} - \rho_{i,2}b_{i,2})$ and $\lambda_i = \beta(1-2\alpha)[\rho_{i,2}(a_{i,2} - b_{i,2}x_1) + (c_{i,2} - \rho_{i,2}b_{i,2})w_1] - \rho_{i,1}a_{i,1}$.

Proposition 3.5 implies that the Hessian matrix of $\Gamma_{i,1}$ is negative semi-definite, and, hence, the problem is a concave quadratic program. Therefore, the KKT conditions above are, again, sufficient for $\vec{u}_1^*$ to be a global optimal solution; and, the above mentioned methods available in Nocedal and Wright [46] and may be used to find it. Next, we focus on the unconstrained solution ($\delta_i^* = 0, \forall i$).

Proposition 3.7 (Uniqueness of the global maximizer and optimality for non-identical users)

(i) Suppose that users are non-identical. Then, the global maximizer of $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$ is unique and given by $u_{1,1}^{**} = \frac{\tilde{\tau}_1}{\tilde{\kappa}}$, $u_{2,1}^{**} = \frac{\tilde{\tau}_2}{\tilde{\kappa}}$ and $u_{k+2,1}^{**} = \hat{\lambda}_{k+2} + \hat{e}_{(k+2,1)}u_{1,1}^{**} + \hat{e}_{(k+2,2)}u_{2,1}^{**}$, for $k = 1, \cdots, n-2$, where

$\tilde{\tau}_1 = [\lambda_1 - \sigma_1\hat{\lambda}_n][\sum_{j=0}^{1} e_{(n,n-j)}\hat{e}_{(n-j,2)}] - [\sigma_1 + \sigma_1\hat{e}_{(n,2)}][\lambda_n - \sum_{j=0}^{1} e_{(n,n-j)}\hat{\lambda}_{n-j}]$,

$\tilde{\tau}_2 = [\epsilon_1 + \sigma_1\hat{e}_{(n,1)}][\lambda_n - \sum_{j=0}^{1} e_{(n,n-j)}\hat{\lambda}_{n-j}] - [\lambda_1 - \sigma_1\hat{\lambda}_n][\sigma_n + \sum_{j=0}^{1} e_{(n,n-j)}\hat{e}_{(n-j,1)}]$,

$\tilde{\kappa} = [\epsilon_1+\sigma_1\hat{e}_{(n,1)}][\sum_{j=0}^{1} e_{(n,n-j)}\hat{e}_{(n-j,2)}]-[\sigma_1+\sigma_1\hat{e}_{(n,2)}][\sigma_n+\sum_{j=0}^{1} e_{(n,n-j)}\hat{e}_{(n-j,1)}]$, $\hat{\lambda}_{k+2}$ *and* $\hat{e}_{(k+2,m)}$ *are as defined before in Proposition 3.3. In addition, we have, for* $i = 1, \cdots, n$, $e_{(i,i)} = \epsilon_i$ *and*

$$e_{(i,j)} = \begin{cases} \sigma_i, & (i,j) \in \{(i,i+1),(1,n)\},\ i = 1,\cdots,n-1 \\ \sigma_i, & (i,j) \in \{(i,i-1),(n,1)\},\ i = 2,\cdots,n \\ 0, & o.w. \end{cases}.$$

(ii) If $0 \le u^{**}_{i,1} \le x_1$, *for all* i, *then* $u^{**}_{i,1}$, *given above, is the optimal solution for the decentralized problem.*

Proof *See Appendix.*

When all users are identical (*i.e.* $\epsilon_i = \epsilon$, $\sigma_i = \sigma$ and $\lambda_i = \lambda$, where $\epsilon, \sigma, \lambda < 0$), it is possible to obtain a compact expression for the Nash equilibrium.

Corollary 3.6 (Unique Nash equilibrium for identical users) *The* $n-$*player game corresponding to the decentralized problem in a ring configuration has a unique Nash equilibrium given by, for all* i,

$$u^*_{i,1} = \begin{cases} \lambda/(2\sigma+\epsilon), & \lambda > (2\sigma+\epsilon)x_1 \\ x_1, & o.w. \end{cases}$$

Proof *See Appendix.*

In a ring configuration with identical users, all users consume the same amount from the aquifer in each period. So long as the cost-revenue structure is such that the condition $\lambda > (2\sigma + \epsilon)x_1$ is satisfied, the water stock is not depleted; otherwise, all users deplete the initial stock in the first period leaving nothing for the next period. We think that this observation may have significant implications for policy makers in setting the unit costs for underground water usage if decentralized decision making is to be employed. Since users' optimal decisions are identical, it may be possible to convince the users either *(i)* into a cooperative game rather than the competitive one they are playing, or *(ii)* into enforcing a centralized decision. In the next section, we take up this important issue of possible coordination through unit prices; that is, whether or not single price mechanisms exist through which the decentralized solution may converge to the centralized optimal decision. Similar to the strip configuration, it is possible to construct the above game with imperfect information about the parameter α by replacing the expressions involving α with their expectation for identical users.

3.3.2 The Centralized Problem

Analogous to the strip configuration, the centralized problem for the ring configuration envisions that a social planner aims at determining the optimal underground water usage for each user so as to maximize the total discounted profit for the entire system stated in Eqn (3.12) subject to Eqn (3.15) where $x_{i,2}$ is characterized by Eqn (3.14). Since the objective function of the optimization is a positive linear combination of the individual profit-to-go functions, we have the following result.

Corollary 3.7 (Myopic optimality, Positivity, Continuity, Concavity)

(i) The myopically optimal water usage in period t is to deplete all stock $[g^*_{i,t}(u_{i,t}, x_{i,t}) = g_{i,t}(x_{i,t}, x_{i,t})]$.

(ii) For a given $\vec{x}_1$, $\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1)$ is strictly increasing in $u_{i,1}$ at $u_{i,1} = 0$ if $\rho_{i,1}a_{i,1} \geq \beta(\rho_{i,2}a_{i,2} + c_{i,2}w_1)$, for all i.

(iii) For a given $\vec{x}_1$, $\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1)$ is continuous and jointly concave in $\vec{u}_1$ if and only if $c_{i,2} \leq \rho_{i,2}b_{i,2}$, for all i.

The above result once again implies that the centralized problem in the ring configuration reduces to an equivalent single period concave quadratic optimization problem subject to the initial constraint set $u_{i,1} \leq x_1$ for all i. Constructing the Lagrange function for this problem in a similar fashion, we observe that the KKT conditions are given by $\tilde{B}\vec{u}^*_1 - \vec{\delta}^{*T} = \tilde{W}$, together with (3.9)-(3.11). Similar to the strip configuration, the method of finding the unconstrained solution for the general case of non-identical users is given below.

Proposition 3.8 (Uniqueness of the global maximizer and optimality for non-identical users)

(i) Suppose that users are non-identical. Then, the global maximizer of $\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1)$ *is unique and given by* $u_{1,1}^{**} = \frac{\tilde{\gamma}_1}{\tilde{\sigma}}$, $u_{2,1}^{**} = \frac{\tilde{\gamma}_2}{\tilde{\sigma}}$, $u_{3,1}^{**} = \frac{\tilde{\epsilon}_2 - \tilde{d}_{2,1}u_{1,1}^{**} - \tilde{d}_{2,2}u_{2,1}^{**}}{\tilde{d}_{2,3}}$, $u_{4,1}^{**} = \frac{\epsilon_1 - d_{1,1}u_{1,1}^{**} - d_{1,2}u_{2,1}^{**} - d_{1,3}u_{3,1}^{**}}{d_{1,4}}$ *and*

$u_{k+2,1}^{**} = \hat{\phi}_{k+2} + \hat{e}_{(k+2,1)}u_{1,1}^{**} + \hat{e}_{(k+2,2)}u_{2,1}^{**} + \hat{e}_{(k+2,3)}u_{3,1}^{**} + \hat{e}_{(k+2,4)}u_{4,1}^{**}$, *for* $k = 3, \cdots, n-2$, *where*

$\tilde{\gamma}_1 = (\tilde{d}_{2,3}\tilde{\epsilon}_3 - \tilde{d}_{3,3}\tilde{\epsilon}_2)(\tilde{d}_{4,2}\tilde{d}_{2,3} - \tilde{d}_{4,3}\tilde{d}_{2,2}) - (\tilde{d}_{2,3}\tilde{\epsilon}_4 - \tilde{d}_{4,3}\tilde{\epsilon}_2)(\tilde{d}_{3,2}\tilde{d}_{2,3} - \tilde{d}_{3,3}\tilde{d}_{2,2})$,

$\tilde{\gamma}_2 = (\tilde{d}_{2,3}\tilde{\epsilon}_4 - \tilde{d}_{4,3}\tilde{\epsilon}_2)(\tilde{d}_{3,1}\tilde{d}_{2,3} - \tilde{d}_{3,3}\tilde{d}_{2,1}) - (\tilde{d}_{2,3}\tilde{\epsilon}_3 - \tilde{d}_{3,3}\tilde{\epsilon}_2)(\tilde{d}_{4,1}\tilde{d}_{2,3} - \tilde{d}_{4,3}\tilde{d}_{2,1})$,

$\tilde{\sigma} = (\tilde{d}_{3,1}\tilde{d}_{2,3} - \tilde{d}_{3,3}\tilde{d}_{2,1})(\tilde{d}_{4,2}\tilde{d}_{2,3} - \tilde{d}_{4,3}\tilde{d}_{2,2}) - (\tilde{d}_{3,2}\tilde{d}_{2,3} - \tilde{d}_{3,3}\tilde{d}_{2,2})(\tilde{d}_{4,1}\tilde{d}_{2,3} - \tilde{d}_{4,3}\tilde{d}_{2,1})$,

$\tilde{\epsilon}_i = d_{1,i}\epsilon_i - d_{i,4}\epsilon_1$, $\tilde{d}_{i,j} = d_{1,4}d_{i,j} - d_{i,4}d_{1,j}$, *for* $i = 2,3,4$ *and* $j = 1,2,3$,

$$d_{1,m} = \begin{cases} e_{(1,m)} + \sum_{j=0}^{1} e_{(1,n-j)}\hat{e}_{(n-j,m)}, & m = 1,2,3 \\ \sum_{j=0}^{1} e_{(1,n-j)}\hat{e}_{(n-j,4)}, & m = 4 \end{cases},$$

$d_{2,m} = e_{(2,m)} + e_{(2,n)}\hat{e}_{(n,m)}$, $m = 1,2,3,4$,

$$d_{3,m} = \begin{cases} e_{(n-1,1)} + \sum_{j=0}^{3} e_{(n-1,n-j)}\hat{e}_{(n-j,1)}, & m = 1 \\ \sum_{j=0}^{3} e_{(n-1,n-j)}\hat{e}_{(n-j,m)}, & m = 2,3,4 \end{cases},$$

$$d_{4,m} = \begin{cases} e_{(n,m)} + \sum_{j=0}^{2} e_{(n,n-j)}\hat{e}_{(n-j,m)}, & m = 1,2 \\ \sum_{j=0}^{2} e_{(n,n-j)}\hat{e}_{(n-j,m)}, & m = 3,4 \end{cases}$$

$$\text{and } \epsilon_m = \begin{cases} \phi_1 - \sum_{j=0}^{1} e_{(1,n-j)}\hat{\phi}_{n-j}, & m = 1 \\ \phi_2 - e_{(2,n)}\hat{\phi}_n, & m = 2 \\ \phi_{n-1} - \sum_{j=0}^{3} e_{(n-1,n-j)}\hat{\phi}_{n-j}, & m = 3 \\ \phi_n - \sum_{j=0}^{3} e_{(n,n-j)}\hat{\phi}_{n-j}, & m = 4 \end{cases}.$$

In addition, we have

$\hat{\phi}_{k+2} = [\phi_k - \sum_{j=1}^{4} e_{(k,k+2-j)}\hat{\phi}_{k+2-j}]/[e_{(k,k+2)}]$,

$\hat{e}_{(k+2,m)} = -[\sum_{j=1}^{4} e_{(k,k+2-j)}\hat{e}_{(k+2-j,m)}]/[e_{(k,k+2)}]$, *for* $m = 1,2,3,4$, *with the conventions* $\hat{\phi}_j = 0$, $\hat{e}_{(j,j)} = 1$, *for* $j = 1,2,3,4$, $\hat{e}_{(i,j)} = 0$, *for* $i,j = 1,2,3,4$, $i \neq j$, *and* $e_{(m,i)} = \hat{e}_{(m,j)} = 0$, *for* $\{i,m\} < 1$ *and* $\{i,m\} > n$ *and* $j = 1,2,3,4$, *where, for* $i = 1,\cdots,n$,

$e_{(i,i)} = (\rho_{i,1}b_{i,1} + c_{i,1}) + \beta(1-2\alpha)^2(\rho_{i,2}b_{i,2} - c_{i,2}) + \beta\alpha^2[(\rho_{i-1,2}b_{i-1,2} - c_{i-1,2}) + (\rho_{i+1,2}b_{i+1,2} - c_{i+1,2})]$,

$e_{(i,i-2)} = \beta\alpha^2(\rho_{i-1,2}b_{i-1,2} - c_{i-1,2})$, $e_{(i,i-1)} = \beta\alpha(1-2\alpha)[(\rho_{i-1,2}b_{i-1,2} - c_{i-1,2}) + (\rho_{i,2}b_{i,2} - c_{i,2})]$,

$e_{(i,i+1)} = \beta\alpha(1-2\alpha)[(\rho_{i,2}b_{i,2} - c_{i,2}) + (\rho_{i+1,2}b_{i+1,2} - c_{i+1,2})]$, $e_{(i,i+2)} = \beta\alpha^2(\rho_{i+1,2}b_{i+1,2} - c_{i+1,2})$,

$e_{(i,j)} = 0$, *elsewhere and* $\phi_i = \rho_{i,1}a_{i,1} - \beta[\alpha(\rho_{i-1,2}a_{i-1,2} + c_{i-1,2}w_1) + (1-2\alpha)(\rho_{i,2}a_{i,2} + c_{1,2}w_1) + \alpha(\rho_{i+1,2}a_{i+1,2} + c_{i+1}w_1)] + \beta[\alpha\rho_{i-1,2}b_{i-1,2}(x_1 + w_1) + (1-2\alpha)\rho_{i,2}b_{i,2}(x_1 + w_1) + \alpha\rho_{i+1,2}b_{i+1,2}(x_1 + w_1)]$.

(ii) If $0 \le u_{i,1}^{**} \le x_1$, *for all* i, *then* $u_{i,1}^{**}$, *given above, is the optimal solution for the centralized problem.*

Proof *See Appendix.*

For identical users, we find that the results for the optimal solution of the centralized ring configuration are *exactly the same* as those for the strip configuration, as stated below.

Corollary 3.8 (Uniqueness of the global maximizer and optimality for identical users)

(i) Suppose that users are identical and $c_2 \le \rho_2 b_2$. *Then, the global maximizer of* $\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1)$ *is unique and given by*

$u_{i,1}^{**} = u^{**} = [\rho_1 a_1 - \beta(\rho_2 a_2 + c_2 w_1) + \beta\rho_2 b_2(x_1 + w_1)]/[(\rho_1 b_1 + c_1) + \beta(\rho_2 b_2 - c_2)]$, $\forall i$

(ii) If $0 \le u_{i,1}^{**} \le x_1$, *for all* i, *then the optimal solution for the centralized problem is given by* u^{**} *above.*

Proof *See Appendix.*

Corollaries 3.4 and 3.8 indicate that the configuration of the users does not change the optimal allocation of water among users when the system is managed centrally. In the strip configuration, we have shown that it is not possible to coordinate the system through a centrally set unit cost (c_t). This was due to the finding that decentralized decisions of users are non-identical even for identical users due to their differing locations over the common aquifer. For the ring configuration, the decentralized optimal solution is the same for all identical users. The next question is: Is it possible to coordinate the system in the ring configuration?

Corollary 3.9 (No coordination) *In a ring configuration with n identical users, for $(\rho_t b_t + c_t)x_0 < \rho_t a_t < (2\rho_t b_t + c_t)x_0$, there does not exist a periodic unit pumpage cost c_t that equates the Nash equilibrium with the centralized optimal solution, for $t = 1, 2$.*

Proof *See Appendix.*

According to the above corollary, under the cost structure adopted herein and by Saak and Peterson [52], the social planner can not entice multiple ($n > 2$) users to behave in accordance with the centralized optimal decision. If the total profits realized from the central allocation of usage are greater than those realized decentrally, then the centralized solution will dominate the decentralized one. Unfortunately, no analytical comparison could be obtained for the total discounted profits realized from the optimal usage quantities under both management systems. However, in the following chapter, we will provide some numerical illustrations and comparisons between the solutions in both strip and ring configurations.

3.4 Double-Layer Ring Configuration

We consider a two-layer ring configuration where each layer (ring) contains n identical users distributed in a circular fashion as in the ring configuration. Figure 3.4 illustrates the hydrology of the aquifer in the double-layer ring configuration. Under this setting, water lateral flows to user i in layer k come from three neighbors. Namely, from users $i-1$ and $i+1$ within layer k and from user i in the other layer, for $i = 1, \cdots, n$ and $k = 1, 2$. Since users are identical, the lateral transmissivity coefficients between adjacent users $(i-1, i, i+1)$ within layer k are the same for all users. More specifically, $\alpha_{(i,k)} = \alpha_k$, for all i and k. Also, the lateral transmissivity coefficient between adjacent users indexed by index i across the two layers is given by α, for $i = 1, \cdots, n$.

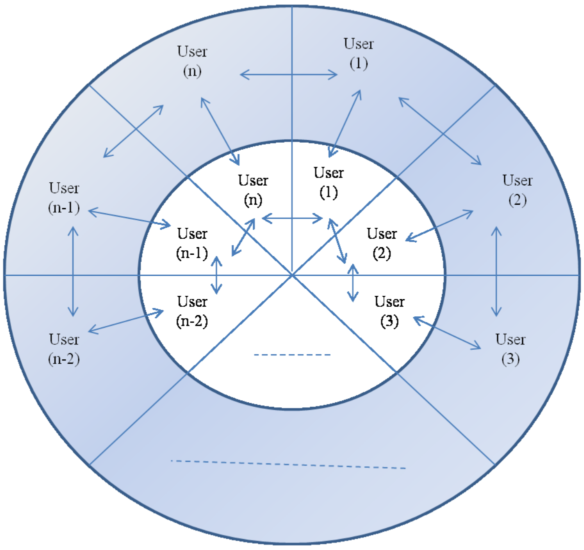


Figure 3.4: Hydrology of the aquifer in the two-layer ring configuration

The water stock level of user (i, k) at the beginning of period $t+1$ is given by

$$x_{(i,k),t+1} = x_{(i,k),t} + w_{(i,k),t} - u_{(i,k),t} + Q_{(i-1,k),(i,k),t} + Q_{(i+1,k),(i,k),t} + Q_{(i,j),(i,k),t} \quad (3.16)$$

where $x_{(i,k),t}$ is the water stock level of user (i, k) at the beginning of period t,

$w_{(i,k),t}$ is the water recharge at user (i,k) in period t, $u_{(i,k),t}$ is the water pumpage by user (i,k) in period t, $Q_{(i-1,k),(i,k),t}$ and $Q_{(i+1,k),(i,k),t}$ are, respectively, the lateral flows from users $(i-1,k)$ and $(i+1,k)$ to user (i,k) in period t and $Q_{(i,j),(i,k),t}$ is the lateral flow from user i in layer j to user i in layer k in period t, $i = 1, \cdots, n$, $j, k = 1, 2,\ j \neq k$ and $t = 1, 2$. Similar to the previous two configurations, we assume an aquifer recharge $w_{(i,k),1}$ for all i and k at the beginning of period 2. Using Darcy's Law and substituting the respective lateral flows in the Eqn (3.16), we obtain

$$x_{(i,k),t+1} = x_{(i,k),t} + w_{(i,k),t} - u_{(i,k),t} - \alpha_k[(x_{(i,k),t} + w_{(i,k),t} - u_{(i,k),t}) - (x_{(i-1,k),t} + w_{(i-1,k),t} - u_{(i-1,k),t})] - \alpha_k[(x_{(i,k),t} + w_{(i,k),t} - u_{(i,k),t}) - (x_{(i+1,k),t} + w_{(i+1,k),t} - u_{(i+1,k),t})] - \alpha[(x_{(i,k),t} + w_{(i,k),t} - u_{(i,k),t}) - (x_{(i,j),t} + w_{(i,j),t} - u_{(i,j),t})].$$

In the analysis below, we assume that initial water stocks $x_{(i,k),1}$ and aquifer recharges $w_{(i,k),1}$ are identical for all users $i = 1, ..., n$ within the same layer; i.e. $x_{(i,k),1} = x_{k,1}$ and $w_{(i,k),1} = w_{k,1}$. Hence, with identical users within layer k, we have $x_{(i,k),t} = x_{k,t}$ and $w_{(i,k),t} = w_{k,t}$. Eventually, in the last equation, for $i = 1, \cdots, n$ and $j, k = 1, 2,\ j \neq k$, $x_{(i,k),t+1}$ reduces to

$$x_{(i,k),t+1} = (1-\alpha)(x_{k,t}+w_{k,t}) + \alpha(x_{j,t}+w_{j,t}) - \hat{\alpha}u_{(i,k),t} - \alpha_k[u_{(i-1,k),t} + u_{(i+1,k),t}] - \alpha u_{(i,j),t} \tag{3.17}$$

where $\hat{\alpha} = 1 - \alpha - 2\alpha_k$ and $(\alpha + 2\alpha_k) \in [0, 0.5]$, for $k = 1, 2$. Furthermore, with identical users within layer k, the profit function of groundwater usage realized by user (i,k) for time period t is given

$$g_{(i,k),t}(u_{(i,k),t}, x_{(i,k),t}) = [\rho_t a_t - c_t(x_{k,0} - x_{k,t})]u_{(i,k),t} - 0.5(\rho_t b_t + c_t)u^2_{(i,k),t} \tag{3.18}$$

where $x_{k,0} = x_{(i,k),0},\ \forall i$ and k, is the initial water stock at each user in layer k and the cost-revenue parameters $\rho_t, a_t, b_t, c_t > 0$ and satisfy the following condition, for $k = 1, 2$ and $t = 1, 2$,

$$(\rho_t b_t + c_t)x_{k,0} < \rho_t a_t < (2\rho_t b_t + c_t)x_{k,0} \tag{3.19}$$

Lemma 3.2 (Positivity, Continuity, Concavity)

(i) For $u_{(i,k),t} \leq x_{(i,k),t} \leq x_{k,0}$, the function $g_{(i,k),t}(u_{(i,k),t}, x_{(i,k),t})$ is strictly increasing in $u_{(i,k),t}$, $i = 1, \cdots, n,\ k = 1, 2,\ t =, 1, 2.$

(ii) The function $g_{(i,k),t}(u_{(i,k),t}, x_{(i,k),t})$ is continuous and concave in $u_{(i,k),t}$, $i = 1, \cdots, n,\ t = 1, 2.$

Proof *See Appendix.*

Below, we present the solution of the decentralized problem corresponding to the this configuration.

3.4.1 The Decentralized Problem

Similar to the decentralized problems of the strip and ring configurations, for $t = 1, 2$, the decentralized problem of the double-layer ring configuration, for $i = 1, \cdots, n$ and $k = 1, 2$, is as follows

$$\Gamma^*_{(i,k),t}(\vec{u}_t, \vec{x}_t) = \max_{u_{(i,k),t}} \Gamma_{(i,k),t}(\vec{u}_t, \vec{x}_t) = \max_{u_{(i,k),t}} [g_{i,t}(u_{(i,k),t}, x_{(i,k),t}) + \beta_{i,t}\Gamma^*_{i,t+1}(\vec{u}_{t+1}, \vec{x}_{t+1})] \tag{3.20}$$

$$s.t. \quad (3.17) \ \ and \ \ 0 \leq u_{(i,k),t} \leq x_{(i,k),t} \tag{3.21}$$

where $\vec{u}_t = (u_{(1,1),t}, \cdots, u_{(n,1),t}, u_{(1,2),t}, \cdots, u_{(n,2),t)})^T$ is a $2n \times 1$ vector of water usage of all users in period t and $\vec{x}_t = (x_{(1,1),t}, \cdots, x_{(n,1),t}, x_{(1,2),t}, \cdots, x_{(n,2),t})^T$ is a $2n \times 1$ vector of initial water stock of all users in period t. We retain all conventions and notations of the decentralized problems of the previous configurations.

Corollary 3.1 holds for this configuration as well and, hence, the within-period profit function $g_{(i,k),t}(u_{(i,k),t}, x_{(i,k),t})$ attains its maximum at $u^*_{(i,k),t} = x_{(i,k),t}$, for $i = 1, \cdots, n$, $k = 1, 2$ and $t = 1, 2$. Hence, in the optimal solution, all users deplete water resources in the very last period ($i.e.$, $u^*_{(i,k),2} = x_{(i,k),2}$, $\forall i$ *and* k). Therefore, we have $\Gamma_{(i,k),1}(\vec{u}_1, \vec{x}_1) = [g_{(i,k),1}(u_{(i,k),1}, x_{(i,k),1}) + \beta g_{(i,k),2}(x_{(i,k),2}, x_{(i,k),2})]$, where $x_{(i,k),2}$ is obtained from Eqn (3.17) when $t = 1$.

Proposition 3.9 (Positivity, Continuity, Concavity)

(i) $\Gamma_{(i,k),1}(\vec{u}_1, \vec{x}_1)$ *is strictly increasing in* $u_{(i,k),1}$ *at* $u_{(i,k),1} = 0$ *if* $\rho_1 a_1 \geq \beta(\rho_2 a_2 + c_2 w_{k,1} - \alpha c_2[(x_{k,1} + w_{k,1}) - (x_{j,1} + w_{j,1})])$, $i = 1, \cdots, n$, $j, k = 1, 2$; $j \neq k$.

(ii) $\Gamma_{(i,k),1}(\vec{u}_1, \vec{x}_1)$ *is continuous and jointly concave in* $\vec{u}_1$ *if and only if* $c_2 \leq \rho_2 b_2$, $i = 1, \cdots, n$, $j, k = 1, 2$; $j \neq k$.

Proof *See Appendix.*

In the sequel, the decentralized problem of this configuration is written as single period problem given by

$$\max_{u_{(i,k),1}} \Gamma_{(i,k),1}(\vec{u}_1, \vec{x}_1) = \max_{u_{(i,k),1}} [g_{(i,k),1}(u_{(i,k),1}, x_{(i,k),1}) + \beta g_{(i,k),2}(x_{(i,k),2}, x_{(i,k),2})] \quad (3.22)$$

$$s.t. \quad 0 < u_{(i,k),1} \leq x_{k,1} \quad (3.23)$$

As the properties of the above problem satisfy those of Theorem 1 in Dasgubta and Maskin [16], we have the existence of a Nash equilibrium as stated below.

Proposition 3.10 (Existence of Nash Equilibrium) *The* $2n-$*player game which corresponds to the decentralized problem in the double-layer ring configuration has (at least one) Nash equilibrium.*

The Nash equilibrium corresponds to the simultaneous solution of $2n$ constrained optimization problems with the constraints $u_{(i,k),1} \leq x_{k,1},\ i = 1,\cdots,n,\ k = 1,2$. Similar to the analysis in the strip and ring configurations, firstly, we investigate the unconstrained solution of the decentralized problem. The FOC corresponding to the objective function of the decentralized problem can be written in a matrix form $B\vec{u}_1^{**} = Z$, where B is a 2×2 symmetric block matrix defined by $B_{2n\times 2n} = \begin{pmatrix} B_1 & B_2 \\ B_2 & B_1 \end{pmatrix}$ and $Z_{2n\times 1} = (\lambda_1 \ \cdots \ \lambda_1 \ \lambda_2 \ \cdots \ \lambda_2)^T$. In matrix B, B_1 is an $n \times n$ matrix given by

$$B_1 = \begin{pmatrix} \gamma & \epsilon & 0 & 0 & \dots & \epsilon \\ \epsilon & \gamma & \epsilon & 0 & \dots & 0 \\ 0 & \epsilon & \gamma & \epsilon & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \epsilon & \gamma & \epsilon \\ \epsilon & 0 & \dots & 0 & \epsilon & \gamma \end{pmatrix}$$

and B_2 is an $n \times n$ diagonal matrix given by $B_2 = \sigma I$, where I is the $n \times n$ identity matrix. The matrix B_1 elements are as follows $\gamma = \beta(1-\alpha-2\alpha_k)^2(c_2-\rho_2 b_2)-(\rho_1 b_1+c_1)$, $\epsilon = \beta\alpha_k(1-\alpha-2\alpha_k)(c_2-\rho_2 b_2)$, $\sigma = \beta\alpha(1-\alpha-2\alpha_k)(c_2-\rho_2 b_2)$ and for $i = 1,\cdots,n$ and $j,k = 1,2,\ j \neq k$, $\lambda_k = \beta(1-\alpha-2\alpha_k)[\rho_2 a_2 - c_2 x_{k,0} - (\rho_2 b_2+c_2)[(1-\alpha)(x_{k,1}+w_{k,1})+\alpha(x_{j,1}+w_{j,1})]]-\rho_1 a_1$.

The system $B\vec{u}_1^{**} = Z$ has a unique solution given by $\vec{u}_1^{**} = B^{-1}Z$, where B^{-1} can be found using Theorem 2.1 in Lu and Shiou [37]. Moreover, we observe that the solution $u^{**}_{(i,k),1} = u^{**}_{k,1}$, for all i and k, satisfies the system $B\vec{u}_1^{**} = Z$ and, hence, it is the unique solution of $B\vec{u}_1^{**} = Z$. The following results gives the Nash equilibrium solution of the decentralized problem.

Proposition 3.11 (Unique Nash equilibrium for identical users within each layer) *The $2n-$player game corresponding to the decentralized problem in a double-layer ring configuration has a unique Nash equilibrium given as follows, for $i = 1,\cdots,n$ and $j,k = 1,2,\ j \neq k$,*

$$u^*_{(i,k),1} = \begin{cases} \frac{(2\epsilon+\gamma)\lambda_k - \sigma\lambda_j}{(2\epsilon+\gamma)^2-\sigma^2}, & 0 < (2\epsilon+\gamma)\lambda_k - \sigma\lambda_j < [(2\epsilon+\gamma)^2-\sigma^2]x_{k,1} \\ x_{k,1}, & o.w. \end{cases}$$

Proof *See Appendix.*

If we assume that all users in the two layers are identical, which means that they have the same initial stock levels as well as the same recharge values in period 1, then the unique solution in the previous result becomes as shown below.

Corollary 3.10 (Unique Nash equilibrium for identical users within the system) *Suppose $x_{k,1} = x_1$ and $w_{k,1} = w_1$, $\forall\ k$. Then, the $2n-$player game corresponding to the decentralized problem in a double-layer ring configuration has a unique Nash equilibrium given as follows, for $i = 1, \cdots, 2n$,*

$$u^*_{i,1} = \begin{cases} \frac{\lambda}{(2\epsilon+\gamma)+\sigma}, & 0 > \lambda > [(2\epsilon+\gamma)+\sigma]x_1 \\ x_1, & o.w. \end{cases}$$

Proof *See Appendix.*

In the next section, we present the solution of the centralized problem corresponding to the double-layer ring configuration.

3.4.2 The Centralized Problem

Analogous to the ring configuration, the centralized problem for this configuration envisions that a social planner aims at determining the optimal underground water usage for each user so as to maximize the total discounted profit for the entire system. For $t = 1, 2$, the centralized problem is given as follows

$$\tilde{\Gamma}^*_t(\vec{u}_t, \vec{x}_t) = \max_{\vec{u}_t} \tilde{\Gamma}_t(\vec{u}_t, \vec{x}_t) = \max_{\vec{u}_t}\{\{\sum_{k=1}^{2}\{\sum_{i=1}^{n} g_{i,t}(u_{(i,k),t}, x_{(i,k),t})\}+\beta_t\tilde{\Gamma}^*_{t+1}(\vec{u}_{t+1}, \vec{x}_{t+1})\} \tag{3.24}$$

$$s.t. \quad (3.21)$$

Since the objective function of the optimization is a positive linear combination of the individual profit-to-go functions, we have the following result.

Corollary 3.11 (Myopic optimality, Positivity, Continuity, Concavity)

(i) The myopically optimal water usage in period t is to deplete all stock $[g^*_{(i,k),t}(u_{(i,k),t}, x_{(i,k),t}) = g_{(i,k),t}(x_{(i,k),t}, x_{(i,k),t})]$, for all i, k and t.

(ii) For a given $\vec{x}_1$, $\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1)$ is strictly increasing in $u_{(i,k),1}$ at $u_{(i,k),1} = 0$ if $\rho_1 a_1 \geq \beta(\rho_2 a_2 + c_2 w_{k,1} - \alpha c_2[(x_{k,1} + w_{k,1}) - (x_{j,1} + w_{j,1})])$, $i = 1, \cdots, n$, $j, k = 1, 2$; $j \neq k$.

(iii) For a given $\vec{x}_1$, $\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1)$ is continuous and jointly concave in $\vec{u}_1$ if and only if $c_2 \leq \rho_2 b_2$, $i = 1, \cdots, n$, $j, k = 1, 2$; $j \neq k$.

According to parts (i) and (ii) of the above result, the centralized problem stated above becomes

$$\max_{\vec{u}_1} \tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1) = \max_{\vec{u}_1} \{\sum_{k=1}^{2} \sum_{i=1}^{n} [g_{(i,k),1}(u_{(i,k),1}, x_{(i,k),1}) + \beta g_{(i,k),2}(x_{(i,k),2}, x_{(i,k),2})]\} \tag{3.25}$$

$$s.t. \quad 0 < u_{(i,k),1} \leq x_{k,1} \tag{3.26}$$

Again, we start with the unconstrained solution of the centralized problem. It is found that the FOC corresponding to the centralized problem can be given by the system $\tilde{B}\vec{u}_1^{**} = \tilde{Z}$, where $\tilde{B}$ is a 2×2 symmetric block matrix defined by

$$\tilde{B}_{2n\times 2n} = \begin{pmatrix} \tilde{B}_1 & \tilde{B}_2 \\ \tilde{B}_2 & \tilde{B}_1 \end{pmatrix} \text{ and } \tilde{Z}_{2n\times 1} = \begin{pmatrix} \theta_1 & \cdots & \theta_1 & \theta_2 & \cdots & \theta_2 \end{pmatrix}^T .$$

In matrix $\tilde{B}$, $\tilde{B}_1$ is an $n \times n$ matrix given by

$$\tilde{B}_{1(n\times n)} = \begin{pmatrix} \phi & \phi_1 & \phi_2 & 0 & 0 & 0 & \dots & \phi_2 & \phi_1 \\ \phi_1 & \phi & \phi_1 & \phi_2 & 0 & 0 & \dots & 0 & \phi_2 \\ \phi_2 & \phi_1 & \phi & \phi_1 & \phi_2 & 0 & \dots & 0 & 0 \\ 0 & \phi_3 & \omega_2 & \omega_1 & \omega_2 & \phi_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \phi_3 & \omega_2 & \omega_1 & \omega_2 & \phi_3 & 0 \\ 0 & 0 & \dots & 0 & \phi_2 & \phi_1 & \phi & \phi_1 & \phi_2 \\ \phi_2 & 0 & \dots & 0 & 0 & \phi_2 & \phi_1 & \phi & \phi_1 \\ \phi_1 & \phi_2 & \dots & 0 & 0 & 0 & \phi_2 & \phi_1 & \phi \end{pmatrix}$$ and $\tilde{B}_2$ is an $n \times n$ matrix given by $$\tilde{B}_{2(n\times n)} = \begin{pmatrix} \eta & \eta_1 & 0 & 0 & \dots & \eta_1 \\ \eta_1 & \eta & \eta_1 & 0 & \dots & 0 \\ 0 & \eta_1 & \eta & \eta_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \eta_1 & \eta & \eta_1 \\ \eta_1 & 0 & \dots & 0 & \eta_1 & \eta \end{pmatrix},$$ where

$\phi = \beta[(1-\alpha-2\alpha_k)^2 + 2\alpha_k^2 + \alpha^2](\rho_2 b_2 - c_2) + (\rho_1 b_1 + c_1)$, $\phi_1 = 2\beta\alpha_k(1-\alpha-2\alpha_k)(\rho_2 b_2 - c_2)$, $\phi_2 = \beta\alpha_k^2(\rho_2 b_2 - c_2)$, $\eta = 2\beta\alpha(1-\alpha-\alpha_1-\alpha_2)(\rho_2 b_2 - c_2)$, $\eta_1 = \beta\alpha(\alpha_1+\alpha_2)(\rho_2 b_2 - c_2)$ and $\theta_k = \rho_1 a_1 - \beta(\rho_2 a_2 + c_2 w_{k,1}) + \beta\alpha(1-2\alpha)c_2[(x_{k,1} + w_{k,1}) - (x_{j,1} + w_{j,1})] + \beta\rho_2 b_2[((1-\alpha)^2+\alpha^2)(x_{k,1}+w_{k,1}) + 2\alpha(1-\alpha)(x_{j,1}+w_{j,1})]$, for $i = 1, \cdots, n$ and $j, k = 1, 2,\ j \neq k$.

Similar to the decentralized problem, the system $\tilde{B}\vec{u}_1^{**} = \tilde{Z}$ has a unique solution given by $\vec{u}_1^{**} = \tilde{B}^{-1}Z$, where $\tilde{B}^{-1}$ can be found using Theorem 2.1 in Lu and Shiou [37]. We observe that the solution $u^{**}_{(i,k),1} = u^{**}_{k,1}$, for all i and k, satisfies the system $\tilde{B}\vec{u}_1^{**} = \tilde{Z}$ and, hence, it is the unique solution of $\tilde{B}\vec{u}_1^{**} = \tilde{Z}$. To facilitate the presentation of the following result, we define

$$\tilde{\theta} = (2\phi_2 + 2\phi_1 + \phi)\theta_k - (2\eta_1 + \eta)\theta_j \text{ and } \tilde{\phi} = (2\phi_2 + 2\phi_1 + \phi)^2 - (2\eta_1 + \eta)^2.$$

The following result gives the unique optimal solution of the centralized problem.

Proposition 3.12 (Uniqueness of the global maximizer and optimality for identical users within each layer) *The global maximizer of $\tilde{\Gamma}_1$ is unique and given by, for $i = 1, \cdots, n$ and $j, k = 1, 2,\ j \neq k$,*

$$u^*_{(i,k),1} = \begin{cases} \tilde{\theta}/\tilde{\phi}, & 0 < \tilde{\theta} < \tilde{\phi} x_{k,1} \\ x_{k,1}, & o.w. \end{cases}$$

Proof *See Appendix.*

When all users in the two layers are identical, which means that they have the same initial stock levels as well as the same recharge values in period 1, then the unique solution in the previous result is as follows.

Corollary 3.12 (Uniqueness of the global maximizer and optimality for identical users within the system) *If $x_{k,1} = x_1$ and $w_{k,1} = w_1$, $\forall\ k$. Then, the global maximizer of $\tilde{\Gamma}_1$ is unique and given by, for $i = 1, \cdots, n$,*

$$u^*_{i,1} = \begin{cases} \frac{\theta}{2(\phi_1+\phi_2+\eta_1)+\phi+\eta}, & 0 > \theta > [2(\phi_1 + \phi_2 + \eta_1) + \phi + \eta] x_1 \\ x_1, & o.w. \end{cases}$$

Proof *See Appendix.*

3.5 Generalization to Multi-Layer Ring Configuration

In the last two sections, we discussed the analysis of the decentralized and centralized problems corresponding to the double-layer ring configuration. We found that with identical users, the solutions of both problems are unique. Herein, we aim to generalize the previous analysis to the case of multi-layer ring configuration, where each layer includes n identical users. Consider a setting of m layers each contains n identical users, where the hydrology of the aquifer under this configuration is depicted in Figure 3.5. As illustrated in Figure 3.5, groundwater at user i in layer k has hydrologic interactions with four adjacent users; two neighbors within the same layer (user $i-1$ and user $i+1$), one user in a lower layer (user i in layer $k-1$) and one user in an upper layer (user i in layer $k+1$). The groundwater lateral transmissivity coefficient between users $i-1$, i and $i+1$ within layer k is the same since users are identical and is denoted by α_k. Likewise, the transmissivity coefficient between user i in layer k and user i in layer $k-1$ is denoted by $\alpha_{(k,k-1)}$ while that between user i in layer k and user i in layer $k+1$ is denoted by $\alpha_{(k,k+1)}$.

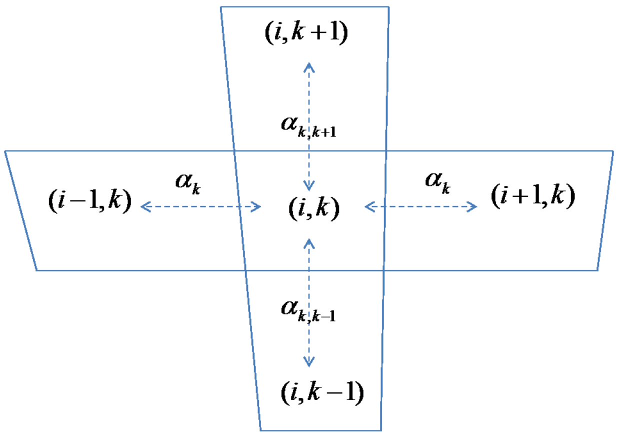


Figure 3.5: Hydrology of the aquifer in the multi-layer ring configuration

The water stock level of user (i,k) at the beginning of period $t+1$ is given by

$$x_{(i,k),t+1} = x_{(i,k),t} + w_{(i,k),t} - u_{(i,k),t} + Q_{(i-1,k),(i,k),t} + Q_{(i+1,k),(i,k),t} + Q_{(i,k-1),(i,k),t} + Q_{(i,k+1),(i,k),t} \tag{3.27}$$

where $x_{(i,k),t}$, $w_{(i,k),t}$, $u_{(i,k),t}$ and the groundwater lateral flows are as defined before in the double-layer ring configuration. Similar to the double-layer ring configuration, in the analysis below, we assume that the initial water stocks $x_{(i,k),1} = x_{k,1}$ and the aquifer recharges $w_{(i,k),1} = w_{k,1}$ for all i and k. Hence, with identical users within layer k, we have $x_{(i,k),t} = x_{k,t}$ and $w_{(i,k),t} = w_{k,t}$. Using Darcy's Law and substituting the respective lateral flows in the Eqn (3.27), we have

$$x_{(i,k),t+1} = (1 - \alpha_{(k,k-1)} - \alpha_{(k,k+1)})(x_{k,t} + w_{k,t}) + \alpha_{(k,k-1)}(x_{k-1,t} + w_{k-1,t}) + \alpha_{(k,k+1)}(x_{k+1,t} + w_{k+1,t}) - (1 - 2\alpha_k - \alpha_{(k,k-1)} - \alpha_{(k,k+1)})u_{(i,k),t} - \alpha_k[u_{(i-1,k),t} + u_{(i+1,k),t}] - \alpha_{(k,k-1)}u_{(i,k-1),t} - \alpha_{(k,k+1)}u_{(i,k+1),t} \tag{3.28}$$

where $(2\alpha_k + \alpha_{(k,k-1)} + \alpha_{(k,k+1)}) \in [0, 0.5]$, for $k = 1, \cdots, m$. As a convention, we set $\alpha_{m+1} \equiv \alpha_0 \equiv 0$. Notice that for $m = 2$, the stock level at the beginning of period 2 in Eqn (3.17) is obtained from Eqn (3.28) when we substitute $\alpha_{(k,k-1)} = \alpha_{(0,1)} = 0$, $k + 1 = j$ and $\alpha_{(k,k+1)} = \alpha$. The profit function of groundwater usage realized by user (i, k) for time period t is as given before in Eqn (3.18) in the double-layer ring configuration.

Below, we discuss the analysis of the decentralized and centralized problems separately.

3.5.1 The Decentralized Problem

Similar to the decentralized problem of the double-layer ring configuration, the multi-layer ring configuration possesses the same decentralized problem as stated in Eqn (3.20) subject to the constraint $0 \leq u_{(i,k),t} \leq x_{(i,k),t}$ and Eqn (3.28), for $t = 1, 2$, $i = 1, \cdots, n$ and $k = 1, \cdots, m$. All other notations and conventions of the decentralized problem in the double-layer ring configuration are retained. Again, Corollary 3.1 holds for this configuration and, hence, the within-period profit function $g_{(i,k),t}(u_{(i,k),t}, x_{(i,k),t})$ attains its maximum at $u^*_{(i,k),t} = x_{(i,k),t}$, for $i = 1, \cdots, n$, $k = 1, \cdots, m$ and $t = 1, 2$. Hence, in the optimal solution, all users deplete water resources in the very last period (*i.e.*, $u^*_{(i,k),2} = x_{(i,k),2}$, $\forall i$ *and* k). Therefore, we have $\Gamma_{(i,k),1}(\vec{u}_1, \vec{x}_1) = [g_{(i,k),1}(u_{(i,k),1}, x_{(i,k),1}) + \beta g_{(i,k),2}(x_{(i,k),2}, x_{(i,k),2})]$, where $x_{(i,k),2}$ is obtained from Eqn (3.28) when $t = 1$.

Proposition 3.13 (Positivity, Continuity, Concavity)

(i) $\Gamma_{(i,k),1}(\vec{u}_1, \vec{x}_1)$ *is strictly increasing in* $u_{(i,k),1}$ *at* $u_{(i,k),1} = 0$ *if* $\rho_1 a_1 \geq \beta(\rho_2 a_2 + c_2 w_{k,1} - \alpha_{(k,k-1)} c_2[(x_{k,1} + w_{k,1}) - (x_{k-1,1} + w_{k-1,1})] - \alpha_{(k,k+1)} c_2[(x_{k,1} + w_{k,1}) - (x_{k+1,1} + w_{k+1,1})])$, $i = 1, \cdots, n,\ k = 1, \cdots, m.$

(ii) $\Gamma_{(i,k),1}(\vec{u}_1, \vec{x}_1)$ *is continuous and jointly concave in* $\vec{u}_1$ *if and only if the characteristic polynomial* $f(\lambda) = \sum_{j=0}^{5} d_j \lambda^j$ *has non-positive eigenvalues,* $i = 1, \cdots, n,\ k = 1, \cdots, m.$

Proof *See Appendix.*

Similar to the double-layer ring configuration, the decentralized problem of this configuration can be written as single period problem as shown before in Eqn (3.22) and Eqn (3.23), where $x_{(i,k),2}$ is obtained from Eqn (3.28) when $t = 1$. If the properties of the above problem (specifically, the joint concavity property in Proposition (ii)) satisfy those of Theorem 1 in Dasgubta and Maskin [16], we have the existence of a Nash equilibrium as stated below.

Proposition 3.14 (Existence of Nash Equilibrium) *The* $mn-$*player game which corresponds to the decentralized problem in the multi-layer ring configuration has (at least one) Nash equilibrium.*

The Nash equilibrium corresponds to the simultaneous solution of mn constrained optimization problems with a single constraint $u_{(i,k),1} \leq x_{k,1},\ i = 1, \cdots, n,\ k = 1, \cdots, m.$ Similar to the analysis in the previous configurations, we start with the unconstrained solution of the decentralized problem. To this end, the FOC of the decentralized problem can be written in a matrix form $B\vec{u}_1^{**} = Z$, where B is an $mn \times mn$ tridiagonal block matrix defined by

$$B_{mn\times mn} = \begin{pmatrix} A_1 & B_1 & 0 & \dots & 0 & 0 \\ C_2 & A_2 & B_2 & \dots & 0 & 0 \\ 0 & C_3 & A_3 & B_3 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & C_{m-2} & A_{m-2} & B_{m-2} & 0 \\ 0 & 0 & \dots & C_{m-1} & A_{m-1} & B_{m-1} \\ 0 & 0 & \dots & 0 & C_m & A_m \end{pmatrix} \text{ and}$$

$$A_k = \begin{pmatrix} \gamma_k & \epsilon_k & 0 & 0 & \dots & \epsilon_k \\ \epsilon_k & \gamma_k & \epsilon_k & 0 & \dots & 0 \\ 0 & \epsilon_k & \gamma_k & \epsilon_k & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \epsilon_k & \gamma_k & \epsilon_k \\ \epsilon_k & 0 & \dots & 0 & \epsilon_k & \gamma_k \end{pmatrix},$$ B_k and C_k are $n \times n$ diagonal matrices given, respectively, by $B_k = \omega_k I$ and $C_k = \sigma_k I$, where I is the $n \times n$ identity matrix and $Z_{mn\times 1} = \ [(\lambda_1 \ \cdots \ \lambda_1) \ (\lambda_2 \ \cdots \ \lambda_2) \ \cdots \ (\lambda_m \ \cdots \ \lambda_m)]^T$.

In the above, $\gamma_k = \beta(1 - 2\alpha_k - \alpha_{(k,k-1)} - \alpha_{(k,k+1)})^2(c_2 - \rho_2 b_2) - (\rho_1 b_1 + c_1)$,

$\epsilon_k = \beta\alpha_k(1 - 2\alpha_k - \alpha_{(k,k-1)} - \alpha_{(k,k+1)})(c_2 - \rho_2 b_2)$,

$\sigma_k = \beta\alpha_{(k,k-1)}(1 - 2\alpha_k - \alpha_{(k,k-1)} - \alpha_{(k,k+1)})(c_2 - \rho_2 b_2)$,

$\omega_k = \beta\alpha_{(k,k+1)}(1 - 2\alpha_k - \alpha_{(k,k-1)} - \alpha_{(k,k+1)})(c_2 - \rho_2 b_2)$ and

$\lambda_k = \beta(1 - 2\alpha_k - \alpha_{(k,k-1)} - \alpha_{(k,k+1)})[\rho_2 a_2 - c_2 x_{k,0} - (\rho_2 b_2 - c_2)[(1 - \alpha_{(k,k-1)} - \alpha_{(k,k+1)})(x_{k,1} + w_{k,1}) + \alpha_{(k,k-1)}(x_{k-1,1} + w_{k-1,1}) + \alpha_{(k,k+1)}(x_{k+1,1} + w_{k+1,1})]] - \rho_1 a_1$, for $k = 1, \cdots, m$.

Similar to the double-layer configuration, we observe that the solution $u^{**}_{(i,k),1} = u^{**}_{k,1}$, for all i and k, satisfies the system $B\vec{u}^{**}_1 = Z$. Therefore, if B is invertible, then the solution $u^{**}_{k,1}$ will be the unique solution of the system $B\vec{u}^{**}_1 = Z$. In the sequel, it is sufficient to solve the system $B\vec{u}^{**}_1 = Z$, where B and Z become

$$B_{m\times m} = \begin{pmatrix} \eta_1 & \omega_1 & 0 & 0 & \dots & 0 \\ \sigma_2 & \eta_2 & \omega_2 & 0 & \dots & 0 \\ 0 & \sigma_3 & \eta_3 & \omega_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \sigma_{m-1} & \eta_{m-1} & \omega_{m-1} \\ 0 & \dots & 0 & 0 & \sigma_m & \eta_m \end{pmatrix}, Z_{m\times 1} = \begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_m \end{pmatrix}^T ,$$

where $\eta_k = 2\epsilon_k + \gamma_k$, for $k = 1, \cdots, m$. We notice that this configuration reduces to the non-identical users strip configuration. The solution of the system $B\vec{u}_1^{**} = Z$ has the same structure of that in Proposition 3.3 in the strip configuration. However, in this case, we have, $m = n$, $e_{(i,i)} = \eta_i$, for $i = 1, \cdots, m$, and

$$e_{(i,j)} = \begin{cases} \omega_i, & (i,j) = (i, i+1),\ i = 1, \cdots, n-1 \\ \sigma_i, & (i,j) = (i, i-1),\ i = 2, \cdots, n \\ 0, & o.w. \end{cases}.$$

The above implies that an m-layer ring configuration, where each layer contains n identical users, reduces to the strip configuration of size m non-identical users along the strip. This is an expected result since within each layer, users pump water equally since they are identical and, hence, the pumpage profile in each layer is only represented by that of one user in the same layer. However, as we move across the layers, starting from layer 1 (which is equivalent to one extreme user of a strip), users pump water in accordance with water usage behavior in the strip configuration of non-identical users, where the second extreme user of the strip is layer m, and the layers between the extremes (indexed from $m = 2, \cdots, m-1$) represent the non-extreme users along the strip.

In the following section, we see that the solution of the centralized problem of the multi-layer ring configuration also reduces to the solution of the centralized problem of the strip configuration with non-identical users.

3.5.2 The Centralized Problem

The centralized problem of this configuration is the same as that given in Eqn (3.24) for the double-layer configuration, but for $k = 1, \cdots, m$. Similar to Corollary 3.11 in the double-ring configuration, we have the following result.

Corollary 3.13 (Myopic optimality, Positivity, Continuity, Concavity)

(i) The myopically optimal water usage in period t is to deplete all stock $[g^*_{(i,k),t}(u_{(i,k),t}, x_{(i,k),t}) = g_{(i,k),t}(x_{(i,k),t}, x_{(i,k),t})]$, for all i, k and t.

(ii) For a given $\vec{x}_1$, $\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1)$ is strictly increasing in $u_{(i,k),1}$ at $u_{(i,k),1} = 0$ if $\rho_1 a_1 \geq \beta(\rho_2 a_2 + c_2 w_{k,1} - \alpha_{k-1} c_2[(x_{k,1} + w_{k,1}) - (x_{k-1,1} + w_{k-1,1})] - \alpha_{k+1} c_2[(x_{k,1} + w_{k,1}) - (x_{k+1,1} + w_{k+1,1})])$, $i = 1, \cdots, n$, $k = 1, \cdots, m$.

(iii) $\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1)$ is continuous and jointly concave in $\vec{u}_1$ if and only if the characteristic polynomial $f(\lambda) = \sum_{j=0}^{5} d_j \lambda^j$ has non-positive eigenvalues, $i = 1, \cdots, n$, $k = 1, \cdots, m$.

In accordance with the first two parts of the above result, the corresponding centralized problem of this configuration is as stated in Eqns (3.25)-(3.26), where $x_{(i,k),2}$ is obtained from Eqn (3.28) when $t = 1$. Analogous to our previous analysis, we start with the unconstrained solution of the centralized problem. Unfortunately, due to the large size of the system, we could not write the FOC of the centralized problem in a neat matrix form. However, we write the FOC as a general linear difference equation, for $i = 1, \cdots, n$ and $k = 1, \cdots, m$, given by

$$\phi_{1,k} u_{(i,k-2),1} + \phi_{2,k} u_{(i,k-1),1} + \phi_{3,k} u_{(i,k),1} + \phi_{4,k} u_{(i,k+1),1} + \phi_{5,k} u_{(i,k+2),1} + \phi_{6,k}[u_{(i-1,k),1} + u_{(i+1,k),1}] + \phi_{7,k}[u_{(i-2,k),1} + u_{(i+2,k),1}] + \phi_{8,k}[u_{(i-1,k-1),1} + u_{(i+1,k-1),1}] + \phi_{9,k}[u_{(i-1,k+1),1} + u_{(i+1,k+1),1}] - \theta_k = 0 \tag{3.29}$$

where $\phi_{1,k} = \beta \alpha_{(k-1,k-2)} \alpha_{(k,k-1)} (\rho_2 b_2 - c_2)$,

$\phi_{2,k} = \beta(\rho_2 b_2 - c_2)\alpha_{(k,k-1)}[(1-2\alpha_k - \alpha_{(k,k-1)} - \alpha_{(k,k+1)}) + (1-2\alpha_{k-1} - \alpha_{(k,k-1)} - \alpha_{(k-1,k-2)})]$,

$\phi_{3,k} = \beta(\rho_2 b_2 - c_2)[2\alpha_k^2 + (1-2\alpha_k - \alpha_{(k,k-1)} - \alpha_{(k,k+1)})^2 + \alpha_{(k,k-1)}^2 + \alpha_{(k,k+1)}^2] + (\rho_1 b_1 + c_1)$,

$\phi_{4,k} = \beta(\rho_2 b_2 - c_2)\alpha_{(k,k+1)}[(1-2\alpha_k - \alpha_{(k,k-1)} - \alpha_{(k,k+1)}) + (1-2\alpha_{k+1} - \alpha_{(k,k+1)} - \alpha_{(k+1,k+2)})]$,

$\phi_{5,k} = \beta\alpha_{(k,k+1)}\alpha_{(k+1,k+2)}(\rho_2 b_2 - c_2)$,

$\phi_{6,k} = 2\beta\alpha_k(1-2\alpha_k - \alpha_{(k,k-1)} - \alpha_{(k,k+1)})(\rho_2 b_2 - c_2)$,

$\phi_{7,k} = \beta\alpha_k^2(\rho_2 b_2 - c_2)$,

$\phi_{8,k} = \beta\alpha_{(k,k-1)}(\alpha_{k-1} + \alpha_k)(\rho_2 b_2 - c_2)$,

$\phi_{9,k} = \beta\alpha_{(k,k+1)}(\alpha_k + \alpha_{k+1})(\rho_2 b_2 - c_2)$ and

$\theta_k = \rho_1 a_1 - \beta(1-\alpha_{(k,k-1)} - \alpha_{(k,k+1)})[\rho_2 a_2 + c_2 w_{k,1} - (\alpha_{(k,k-1)} + \alpha_{(k,k+1)})c_2(x_{k,1} + w_{k,1}) + \alpha_{(k,k-1)}c_2(x_{k-1,1} + w_{k-1,1}) + \alpha_{(k,k+1)}c_2(x_{k+1,1} + w_{k+1,1})] + \beta(1-\alpha_{(k,k-1)} - \alpha_{(k,k+1)})\rho_2 b_2[(1-\alpha_{(k,k-1)} - \alpha_{(k,k+1)})(x_{k,1} + w_{k,1}) + \alpha_{(k,k-1)}(x_{k-1,1} + w_{k-1,1}) + \alpha_{(k,k+1)}(x_{k+1,1} + w_{k+1,1})] - \beta\alpha_{(k,k-1)}[\rho_2 a_2 + c_2 w_{k,1} - (\alpha_{(k,k-1)} + \alpha_{(k-1,k-2)})c_2(x_{k-1,1} + w_{k-1,1}) + \alpha_{(k,k-1)}c_2(x_{k,1} + w_{k,1}) + \alpha_{(k-1,k-2)}c_2(x_{k-2,1} + w_{k-2,1})] - \beta\alpha_{(k,k+1)}[\rho_2 a_2 + c_2 w_{k,1} - (\alpha_{(k,k+1)} + \alpha_{(k+1,k+2)})\ c_2(x_{k+1,1} + w_{k+1,1}) + \alpha_{(k,k+1)}c_2(x_{k,1} + w_{k,1}) + \alpha_{(k+1,k+2)}c_2\ (x_{k+2,1} + w_{k+2,1})] + \beta\alpha_{(k,k-1)}\rho_2 b_2[(1-\alpha_{(k,k-1)} - \alpha_{(k-1,k-2)})(x_{k-1,1} + w_{k-1,1}) + \alpha_{(k,k-1)}(x_{k,1} + w_{k,1}) + \alpha_{(k-1,k-2)}(x_{k-2,1} + w_{k-2,1})] + \beta\alpha_{(k,k+1)}\rho_2 b_2[(1-\alpha_{(k,k+1)} - \alpha_{(k+1,k+2)})(x_{k+1,1} + w_{k+1,1}) + \alpha_{(k,k+1)}(x_{k,1} + w_{k,1}) + \alpha_{(k+1,k+2)}(x_{k+2,1} + w_{k+2,1})]$, for $k = 1, \cdots, m$, with the convention $\alpha_{(0,1)} = \alpha_{(m,m+1)} \equiv 0$.

Similar to the double-layer configuration, we observe that the solution $u^{**}_{(i,k),1} = u^{**}_{k,1}$, for all i and k, satisfies the FOC given above. Therefore, if the parameters are carefully chosen such that the solution of the FOC is unique, then $u^{**}_{k,1}$ will be the unique solution. The corresponding FOC can be written in a matrix form given by the system $\tilde{A}\vec{u}_1^{**} = \tilde{W}$, similar to the strip configuration's FOC corresponding to its centralized problem with non-identical user.

More specifically, $\tilde{A}$ is an $m \times m$ matrix is defined by

$$\tilde{A}_{m\times m} = \begin{pmatrix} \tilde{\phi}_{3,1} & \tilde{\phi}_{4,1} & \phi_{5,1} & 0 & 0 & 0 & \dots & 0 \\ \tilde{\phi}_{2,2} & \tilde{\phi}_{3,2} & \tilde{\phi}_{4,2} & \phi_{5,2} & 0 & 0 & \dots & 0 \\ \phi_{1,3} & \tilde{\phi}_{2,3} & \tilde{\phi}_{3,3} & \tilde{\phi}_{4,3} & \phi_{5,3} & 0 & \dots & 0 \\ 0 & \phi_{1,4} & \tilde{\phi}_{2,4} & \tilde{\phi}_{3,4} & \tilde{\phi}_{4,4} & \phi_{5,4} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \phi_{1,m-3} & \tilde{\phi}_{2,m-3} & \tilde{\phi}_{3,m-3} & \tilde{\phi}_{4,m-3} & \phi_{5,m-3} & 0 \\ 0 & \dots & 0 & \phi_{1,m-2} & \tilde{\phi}_{2,m-2} & \tilde{\phi}_{3,m-2} & \tilde{\phi}_{4,m-2} & \phi_{5,m-2} \\ 0 & \dots & 0 & 0 & \phi_{1,m-1} & \tilde{\phi}_{2,m-1} & \tilde{\phi}_{3,m-1} & \tilde{\phi}_{2,m-1} \\ 0 & \dots & 0 & 0 & 0 & \phi_{1,m} & \tilde{\phi}_{2,m} & \tilde{\phi}_{3,m} \end{pmatrix}$$

and

$\tilde{W}_{m\times 1} = \begin{pmatrix} \theta_1 & \theta_2 & \dots & \theta_{m-1} & \theta_m \end{pmatrix}^T$, where $\tilde{\phi}_{2,k} = \phi_{2,k} + 2\phi_{8,k}$, $\tilde{\phi}_{3,k} = \phi_{3,k} + 2\phi_{6,k} + 2\phi_{7,k}$ and $\tilde{\phi}_{4,k} = \phi_{4,k} + 2\phi_{9,k}$.

The solution of the system $\tilde{A}\vec{u}_1^{**} = \tilde{W}$ has the same structure of that in Proposition 3.4 of the centralized problem of the strip configuration with non-identical users. However, in this case, we have $m = n$, $e_{(i,i)} = \tilde{\phi}_{3,k},\ i = 1, \cdots, m$, and

$$e_{(i,j)} = \begin{cases} \tilde{\phi}_{4,k}, & (i,j) \in \{(1,2),(m,m-1),(2,3),(m-1,m-2)\} \\ \phi_{5,k}, & (i,j) \in \{(1,3),(m,m-2),(2,4),(m-1,m-3)\} \\ \tilde{\phi}_{2,k}, & (i,j) \in \{(2,1),(m-1,m)\} \\ \phi_{1,k}, & (i,j) = (k,k-2) \\ \tilde{\phi}_{2,k}, & (i,j) = (k,k-1) \\ \tilde{\phi}_{4,k}, & (i,j) = (k,k+1) \\ \phi_{5,k}, & (i,j) = (k,k+2) \\ 0, & o.w. \end{cases}$$

for $k = 3, \cdots, m-2$.

3.6 Grid Configuration

In this section, we consider the system of $n-$identical users, each having an equal area (cell), distributed adjacently on a grid over the common groundwater aquifer. The grid configuration might be considered a commonly used configuration from a practical point of view. In particular, large acreage areas (farms) are usually divided between users, lying on the common groundwater aquifer, into smaller acreage areas to be utilized by those users. The main acreage area can be represented, up on division, by a grid having L-rows and K-columns, where L and K are not necessarily equal. Therefore, the total number of users equals the total number of cells in the grid; $n = L \times K$. Figure 3.6 depicts the aquifer's hydrology among users in the grid configuration, where the pumpage quantity of user (l,k) in period t is indicated by $u_{(l,k),t}$, $l = 1, \cdots, L$, $k = 1, \cdots, K$ and $t = 1, 2$. Notice that for $L = 1$ and $K \geq 2$, we have $n \geq 2$, which corresponds to the strip configuration of size n. Also, for $L = 2$ and $K = 2$, we have $n = 4$, which corresponds to the ring configuration with four users. Grids having $L \geq 3$, $K \geq 2$, and, hence, $n \geq 6$ users are considered in this analysis. In this section, square grids having the same number of rows and columns are considered. Therefore, the total number of users on the grid is ($n = L^2$). All other notations and conventions regarding the water usage profit function given in Eqn (3.1) and its parameters given in Eqn (3.2) are retained. Notice here that users have the same time-variant cost-revenue parameters. Also, we assume the equality of users initial water stock values among users and we assume no aquifer recharge in this configuration.

As will be shown in the next sections, we could write the FOC corresponding to the decentralized problem in a matrix from. But, we could not obtain the analytical solution from the FOC. As will be supported by our numerical study in the next chapter, we could obtain some conjectures on the number of distinct solutions of the FOC for both odd and even square grids. On the other hand, unfortunately, we neither could write the FOC corresponding to the centralized problem nor we could obtain an analytical centralized solution. Again, as will be supported by the numerical study in the next chapter, we state, as a conjecture, that the centralized solution is independent of the transmissivity coefficient, α, and it results in the same centralized solutions in the strip and ring configurations.

$u_{(1,1),t}$	$u_{(1,2),t}$	$\cdots\cdots$	$u_{(1,K-1),t}$	$u_{(1,K),t}$
$u_{(2,1),t}$	$u_{(2,2),t}$	$\cdots\cdots$	$u_{(2,K-1),t}$	$u_{(2,K),t}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$u_{(L-1,1),t}$	$u_{(L-1,2),t}$	$\cdots\cdots$	$u_{(L-1,K-1),t}$	$u_{(L-1,K),t}$
$u_{(L,1),t}$	$u_{(L,2),t}$	$\cdots\cdots$	$u_{(L,K-1),t}$	$u_{(L,K),t}$

Figure 3.6: Hydrology of the aquifer in the grid configuration

The general grid structure depicted above demonstrates that the grid can be divided into three disjoint sets of users according to their hydrologic dynamics governed by Darcy's Law. Namely, let S_1 be the set of the decision variables of the pumpage quantities in period t corresponding to all users who lie on the grid corners, we call them the *corner users*, where each user in S_1 has two neighbors. The second set is S_2, which contains the decision variables of the pumpage quantities in period t corresponding to all users lying on the grid outer edges, we call them the *edge users*, where each user in S_2 has three neighbors. The third set, S_3, is the set of the decision variables of the pumpage quantities in period t corresponding to the internal users on the grid, we call them the *internal users*, where each user in S_3 has four neighbors.

If L is even, the grid has L^2 identical users distributed equally into four quadrants (numbered in a clockwise fashion) with respect to the the center of the grid. Namely, an even grid can be divided into four identical quadrants, q_i, $i = 1, 2, 3, 4$, each having $\frac{L^2}{4}$ identical users. Each cell in the grid represents a user having a decision variable in period t denoted by $u_{(i,j,k),t}$, where $i = 1, 2, 3, 4$ denotes the quadrant index, j denotes the horizontal coordinate (the horizontal distance from the grid's center) and k denotes the vertical coordinate (the vertical distance from the grid's center), for $j, k = 1, \cdots, \frac{L}{2}$, $t = 1, 2$. Figure 3.7a illustrates the grid structure for $L = 4$. Obviously, the four quadrants are identical and symmetric in terms of the number of users each has and the nature of hydrologic dynamics among users within each quadrant (the number of adjacent users each user interacts with).

On the other hand, if L is odd, the center of the grid is the user occupying the grid's central cell. Namely, by considering the general grid structure in Figure 3.6, that user has the cell number $\frac{L^2+1}{2}$. Let $u_{(0,0),t}$ denote the decision variable corresponding to that user. Also, let $u_{(j,0),t}$ denote the decision variables corresponding to the users on the central horizontal strip of the grid, $j = \pm 1, \cdots, \pm\frac{L-1}{2}$, where $(+j)$ is the user's horizontal coordinate right to $u_{(0,0),t}$ and $(-j)$ is the user's horizontal coordinate left to $u_{(0,0),t}$. Similarly, the decision variables of users on the central vertical strip are denoted by $u_{(0,k),t}$, $k = \pm 1, \cdots, \pm\frac{L-1}{2}$, where $(+k)$ is the user's vertical coordinate above $u_{(0,0),t}$ and $(-k)$ is the user's vertical coordinate below $u_{(0,0),t}$. Totally, the central strips' users compose $2(L-1)$ of the total number of users on the grid. The remaining number of users, which is $L^2 - 2L + 1$, will be distributed equally into the four identical and symmetric quadrants of the grid as in the even case, where their corresponding decision variables are denoted by $u_{(i,j,k),t}$, $i = 1, 2, 3, 4$, and $j, k = 1, \cdots, \frac{L-1}{2}$, $t = 1, 2$. Figure 3.7b illustrates the grid structure for $L = 5$.

Quadrant 4 Quadrant 1

$u_{(4,2,2),t}$	$u_{(4,1,2),t}$	$u_{(1,1,2),t}$	$u_{(1,2,2),t}$
$u_{(4,2,1),t}$	$u_{(4,1,1),t}$	$u_{(1,1,1),t}$	$u_{(1,2,1),t}$
$u_{(3,2,1),t}$	$u_{(3,1,1),t}$	$u_{(2,1,1),t}$	$u_{(2,2,1),t}$
$u_{(3,2,2),t}$	$u_{(3,1,2),t}$	$u_{(2,1,2),t}$	$u_{(2,2,2),t}$

Quadrant 3 Quadrant 2

(a) A 4×4 grid structure

Quadrant 4 Quadrant 1

$u_{(4,2,2),t}$	$u_{(4,1,2),t}$	$u_{(0,2),t}$	$u_{(1,1,2),t}$	$u_{(1,2,2),t}$
$u_{(4,2,1),t}$	$u_{(4,1,1),t}$	$u_{(0,1),t}$	$u_{(1,1,1),t}$	$u_{(1,2,1),t}$
$u_{(-2,0),t}$	$u_{(-1,0),t}$	$u_{(0,0),t}$	$u_{(1,0),t}$	$u_{(2,0),t}$
$u_{(3,2,1),t}$	$u_{(3,1,1),t}$	$u_{(0,-1),t}$	$u_{(2,1,1),t}$	$u_{(2,2,1),t}$
$u_{(3,2,2),t}$	$u_{(3,1,2),t}$	$u_{(0,-2),t}$	$u_{(2,1,2),t}$	$u_{(2,2,2),t}$

Quadrant 3 Quadrant 2

(b) A 5×5 grid structure

Figure 3.7: Illustrative examples of even and odd grid structures

Now, we are ready to express the water stock level of each user in the grid at the beginning of period $t + 1$. We first consider even square grids. In period t, for $i = 1, 2, 3, 4$, we have $S_1 = \{u_{(i,\frac{L}{2},\frac{L}{2}),t}\}$, where the groundwater stock level at the beginning of period $t + 1$ is given by

$$x_{(i,\frac{L}{2},\frac{L}{2}),t+1} = x_{(i,\frac{L}{2},\frac{L}{2}),t} - (1-2\alpha)u_{(i,\frac{L}{2},\frac{L}{2}),t} - \alpha[u_{(i,\frac{L}{2},\frac{L-2}{2}),t} + u_{(i,\frac{L-2}{2},\frac{L}{2}),t}] \quad (3.30)$$

Also, we have $S_2 = \{u_{(i,j,\frac{L}{2}),t}, u_{(i,\frac{L}{2},k),t} : \ j,k = 1,\cdots,\frac{L-2}{2}\}$, $i = 1,2,3,4$ and $j = 1,\cdots,\frac{L-2}{2}$, where the groundwater stock level at the beginning of period $t+1$ is given by

$$x_{(i,j,\frac{L}{2}),t+1} = x_{(i,j,\frac{L}{2}),t} - (1-3\alpha)u_{(i,j,\frac{L}{2}),t} - \alpha[u_{(i,j-1,\frac{L}{2}),t} + u_{(i,j,\frac{L-2}{2}),t} + u_{(i,j+1,\frac{L}{2}),t}] \quad (3.31)$$

if the three adjacent users to user $(i,j,\frac{L}{2})$ are in the same quadrant. However, if user $(i,j,\frac{L}{2})$ has an adjacent user in another quadrant, Eqn (3.31) should be corrected in its last term which is multiplied by α through adjusting the index of the user located in neighbor quadrant. For user $(i,\frac{L}{2},k) \in S_2$, $i = 1,2,3,4$ and $k = 1,\cdots,\frac{L-2}{2}$, the groundwater stock level at the beginning of period $t+1$ is given by

$$x_{(i,\frac{L}{2},k),t+1} = x_{(i,\frac{L}{2},k),t} - (1-3\alpha)u_{(i,\frac{L}{2},k),t} - \alpha[u_{(i,\frac{L}{2},k-1),t} + u_{(i,\frac{L-2}{2},k),t} + u_{(i,\frac{L}{2},k+1),t}] \quad (3.32)$$

if the three adjacent users to user $(i,\frac{L}{2},k)$ are in the same quadrant. However, if user $(i,\frac{L}{2},k)$ has an adjacent user in another quadrant, Eqn (3.32) should be corrected in its last term which is multiplied by α through adjusting the index of the user located in neighbor quadrant. The last set is given by $S_3 = \{u_{(i,j,k),t} : \ j,k = 1,\cdots,\frac{L-2}{2}\}$, $i = 1,2,3,4$ and $j,k = 1,\cdots,\frac{L-2}{2}$, where the stock level at the beginning of period $t+1$ is given by

$$x_{(i,j,k),t+1} = x_{(i,j,k),t} - (1-4\alpha)u_{(i,j,k),t} - \alpha[u_{(i,j-1,k),t} + u_{(i,j+1,k),t} + u_{(i,j,k-1),t} + u_{(i,j,k+1),t}] \quad (3.33)$$

if the four adjacent users to user (i, j, k) are in the same quadrant. However, if user (i, j, k) has some adjacent users in other quadrants, Eqn (3.33) should be corrected in its last term which is multiplied by α through adjusting the index of user(s) located in neighbor quadrant(s).

In square odd grids, for $i = 1, 2, 3, 4$, in period t, we have $S_1 = \{u_{(i,\frac{L-1}{2}\frac{L-1}{2}),t}\}$, $S_2 = \{u_{(0,\pm\frac{L-1}{2}),t}, u_{(\pm\frac{L-1}{2},0),t}, u_{(i,j,\frac{L-1}{2}),t}, u_{(i,\frac{L-1}{2},k),t} \ : \ j, k \ : \ 1, \cdots, \frac{L-3}{2}\}$ and $S_3 = \{u_{(0,0),t}, u_{(0,\pm\frac{L-3}{2}),t}, u_{(\pm\frac{L-3}{2},0),t}, u_{(i,j,k),t} \ : \ j, k = 1, \cdots, \frac{L-3}{2}\}$. Eqns (3.30)-(3.33) apply for square odd grids as well. However, for the central (horizontal and vertical) strip users and for the central user $(0, 0)$ in S_2 and S_3, respectively, Eqns (3.31)-(3.33) should be corrected according to their new indices.

In this configuration, we consider below the two kinds of decision making - decentralized and centralized problems.

3.6.1 The Decentralized Problem

The decentralized problem of the grid configuration is formulated similar to the previous configurations. For simplicity, we define a general decision variable $u_{i,t}$ to denote the pumpage quantity of water in period t by user i and a general state variable $x_{i,t}$ to denote the water stock level of user i at the beginning of period t, $i = 1, \cdots, n$ and $t = 1, 2$. For $t = 1, 2$, the decentralized problem of of user i in the grid, $i = 1, \cdots, n$, given by

$$\Gamma^*_{i,t}(\vec{u}_t, \vec{x}_t) = \max_{u_{i,t}} \Gamma_{i,t}(\vec{u}_t, \vec{x}_t) = \max_{u_{i,t}} [g_{i,t}(u_{i,t}, x_{i,t}) + \beta_{i,t}\Gamma^*_{i,t+1}(\vec{u}_{t+1}, \vec{x}_{t+1})] \quad (3.34)$$

$$s.t. \quad x_{i,t+1} = \begin{cases} x_{i,t} - (1 - 2\alpha)u_{i,t} - \alpha[u_{j,t} + u_{k,t}], & if \ u_{i,t} \in S_1 \\ x_{i,t} - (1 - 3\alpha)u_{i,t} - \alpha[u_{j,t} + u_{k,t} + u_{l,t}], & if \ u_{i,t} \in S_2 \\ x_{i,t} - (1 - 4\alpha)u_{i,t} - \alpha[u_{j,t} + u_{k,t} + u_{l,t} + u_{m,t}], & if \ u_{i,t} \in S_3 \end{cases} \quad (3.35)$$

$$0 \leq u_{i,t} \leq x_{i,t} \quad (3.36)$$

where the indices j, k, l, m are for users adjacent to user i. In our formulation, the profit function $g_{i,t}(u_{i,t}, x_{i,t})$ is assumed to have the form given in Eqn (3.18) in the double-layer configuration. Also, we assume the same hydrological transmissivity coefficient α across the grid for all users and all periods and, as a convention, we have $\Gamma^*_{i,3}(\vec{u}_3, \vec{x}_3) \equiv 0$ for all $\vec{x}_3$ and $\vec{u}_3$, for all $i = 1, \cdots, n$. Also, we assume that $x_{i,1}$ is normalized to unity for all i; i.e. $x_{i,1} = 1, \ \forall i$. Corollary 3.1 holds for this configuration and, hence, the within-period profit function $g_{i,t}(u_{i,t}, x_{i,t})$ attains its maximum at $u^*_{i,t} = x_{i,t}$, for $i = 1, \cdots, n$ and $t = 1, 2$. Hence, in the optimal solution, all users deplete water resources in the very last period (*i.e.*, $u^*_{i,2} = x_{i,2}, \ \forall i$). Therefore, we have $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1) = [g_{i,1}(u_{i,1}, x_{i,1}) + \beta g_{i,2}(x_{i,2}, x_{i,2})]$, where $x_{(i,k),2}$ is obtained from Eqn (3.35) when $t = 1$.

Proposition 3.15 (Positivity, Continuity, Concavity)

(*i*) $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$ *is strictly increasing in* $u_{i,1}$ *at* $u_{i,1} = 0$ *if* $\rho_1 a_1 \geq \beta(\rho_2 a_2 + c_2 w_1)$ $i = 1, \cdots, n$.

(*ii*) *For each corner user on the grid,* $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$ *is continuous and jointly concave in* $\vec{u}_1$ *if and only if* $c_2 \leq \rho_2 b_2$, $i = 1, \cdots, 4$.

(*iii*) *For each edge user on the grid,* $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$ *is continuous and jointly concave in* $\vec{u}_1$ *if and only if* $c_2 \leq \rho_2 b_2$, $i = 1, \cdots, n_e$, *where* n_e *is the number of edge users on the grid.*

(*iv*) *For each internal user on the grid,* $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$ *is continuous and jointly concave in* $\vec{u}_1$ *if and only if the roots of the function* $\lambda^5 + \gamma_0\lambda^4 + \gamma_1\lambda^3 + \gamma_2\lambda^2 + \gamma_3\lambda + \gamma_4 = 0$ *are all non-positive,* $i = 1, \cdots, n_i$, *where* n_i *is the number of internal users on the grid. The function parameters are given by* $\gamma_0 = [\alpha^4 e_2 - \tilde{\alpha}]e_2$, $\gamma_1 = [\alpha^6 e_2^2 - \alpha^4(\tilde{\alpha}^2 + \alpha^2)e_2 + \alpha\tilde{\alpha}(3 - 4\alpha\tilde{\alpha})]e_2$, $\gamma_2 = [-(3\alpha^8 + \alpha^6\tilde{\alpha}^4)e_2^2 + (4\alpha^2 + 12\alpha^4\tilde{\alpha}^2)e_2 - (6\alpha^4 + 6\alpha^3\tilde{\alpha} + 3\alpha\tilde{\alpha}^3)]e_2^2$, $\gamma_3 = [6\alpha^8 e_2^3 + \alpha^7(\alpha^3 + 3\tilde{\alpha}^2)e_2^2 + \alpha^2(12\alpha^3\tilde{\alpha}^2 - 4\tilde{\alpha} - \alpha^6)e_2 + \alpha^3(3\alpha\tilde{\alpha}(2 - 7\alpha) + \tilde{\alpha}^2(2 - 24\alpha) + 3\alpha^3)]e_2^3$ *and* $\gamma_4 = [\alpha^8\tilde{\alpha}^2(1 - e_2^5) + \alpha^6\tilde{\alpha}^2(-4 + 12\alpha - 26\alpha^2)e_2^4 + \alpha^4\tilde{\alpha}^2(-3 + 16\alpha - 3\alpha^2)e_2^3]e_2$, *where* $e_2 = (c_2 - \rho_2 b_2)$ *and* $\tilde{\alpha} = (1 - 4\alpha)$.

Proof *See Appendix.*

We can now re-state the $n-$user, two-period decentralized problem as follows. For $i = 1, \cdots, n$,

$$\max_{u_{i,1}} \Gamma_{i,1}(\vec{u}_1, \vec{x}_1) = \max_{u_{i,1}} [g_{i,1}(u_{i,1}, 1) + \beta g_{i,2}(x_{i,2}, x_{i,2})] \tag{3.37}$$

$$s.t. \quad 0 < u_{i,1} \leq 1 \tag{3.38}$$

where $x_{i,2}$ is given in Eqn (3.35). Notice that the first part of Eqn (3.35) is equivalent to Eqn (3.30) while its second part is equivalent to Eqns (3.31)-(3.32) whereas its last part is equivalent to Eqn (3.33). At this point, we note that the problem stated in Eqns (3.35), (3.37) and (3.38) corresponds to a single period strategic form- game given by the payoff function $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$ and the strategy set $u_{i,1}$. The strategy set is nonempty, continuous, convex and compact (closed and bounded) and that the payoff function is continuous and jointly concave in the players' strategies, as denoted by Proposition 3.15. Then, from Theorem 1 in Dasgubta and Maskin [16], we have the following result.

Proposition 3.16 (Existence of Nash Equilibrium) *The $n-$player game which corresponds to the decentralized problem in the grid configuration has (at least one) Nash equilibrium.*

Analogous to the previous configurations, we first consider the unconstrained solution of the decentralized problem. To facilitate our analysis, we define

$$z_{i,2}(y) = \frac{\partial}{\partial u_{i,2}} g_{i,2}(u_{i,2}, x_{i,2})|_{(u_{i,2}=x_{i,2}=y)} + \frac{\partial}{\partial x_{i,2}} g_{i,2}(u_{i,2}, x_{i,2})|_{(u_{i,2}=x_{i,2}=y)}$$

as the sum of partial derivatives of $g_{i,2}(u_{i,2}, x_{i,2})$ with respect to $u_{i,2}$ and $x_{i,2}$, respectively, evaluated at $u_{i,2} = x_{i,2} = y$. Then, the FOC of the decentralized problem for user $(i, j, k) \in S_m$, $i = 1, 2, 3, 4$, $j, k = 1, \cdots, \frac{L}{2}(even), \frac{L-1}{2}(odd)$, and $m = 1, 2, 3$ are given by

$$\frac{\partial}{\partial u_{(i,j,k),1}} g_{i,1}(u_{(i,j,k),1}, x_{(i,j,k),1}) - \beta[(1 - (m+1)\alpha] z_i(x_{(i,j,k),2}) = 0 \tag{3.39}$$

where $u_{(i,j,k),1} = u_{(i,\frac{L}{2},\frac{L}{2}),1}$, $x_{(i,j,k),2} = x_{(i,\frac{L}{2},\frac{L}{2}),2}$ (Eqn (3.30)) if $m = 1$, whereas $u_{(i,j,k),1} = u_{(i,j,\frac{L}{2}),1}$, $x_{(i,j,k),2} = x_{(i,j,\frac{L}{2}),2}$ (Eqn (3.31)) if $m = 2$, for $j = 1, ..., \frac{L-2}{2}$, while $u_{(i,j,k),1} = u_{(i,\frac{L}{2},k),1}$, $x_{(i,j,k),2} = x_{(i,\frac{L}{2},k),2}$ (Eqn (3.32)) if $m = 2$, for $k = 1, ..., \frac{L-2}{2}$, and for $m = 3$, $x_{(i,j,k)}$ is as defined before in Eqn (3.33).

By evaluating the FOC in equation (3.39) for a square even $L \times L$ grid, we have $(L-2)^2$ equations for internal users having the form

$$\phi u_{(i,j,k),1} + \sigma \Sigma_{(i',j',k') \in S_3'} u_{(i',j',k'),1} = \lambda \tag{3.40}$$

where S_3' is the set of users (i', j', k') adjacent to internal user (i, j, k). We have $4(L-2)$ equations for edge users having the form

$$\delta u_{(i,j,k),1} + \epsilon \Sigma_{(i',j',k') \in S_2'} u_{(i',j',k'),1} = \eta \tag{3.41}$$

where S_2' is the set of users (i', j', k') adjacent to edge user (i, j, k). Finally, we have four equations for corner users having the form

$$\theta u_{(i,\frac{L}{2},\frac{L}{2}),1} + \omega[u_{(i,\frac{L}{2},\frac{L-2}{2}),1} + u_{(i,\frac{L-2}{2},\frac{L}{2}),1}] = \gamma \tag{3.42}$$

where $\phi = \beta(c_2 - \rho_2 b_2)(1-4\alpha)^2 - (\rho_1 b_1 + c_1)$, $\delta = \beta(c_2 - \rho_2 b_2)(1-3\alpha)^2 - (\rho_1 b_1 + c_1)$, $\theta = \beta(c_2 - \rho_2 b_2)(1-2\alpha)^2 - (\rho_1 b_1 + c_1)$, $\sigma = \beta(c_2 - \rho_2 b_2)\alpha(1-4\alpha)$, $\epsilon = \beta(c_2 - \rho_2 b_2)\alpha(1-3\alpha)$, $\omega = \beta(c_2 - \rho_2 b_2)\alpha(1-2\alpha)$, $\lambda = \beta\rho_2(a_2 - b_2)(1-4\alpha)^2 - \rho_1 a_1$, $\eta = \beta\rho_2(a_2 - b_2)(1-3\alpha)^2 - \rho_1 a_1$ and $\gamma = \beta\rho(a_2 - b_2)(1-2\alpha)^2 - \rho_1 a_1$.

The FOC for a square even $L \times L$ grid can be written in a general matrix form $D^e \vec{u}_1^e = \vec{Y}^e$, where $D^e \in \Re^{L^2 \times L^2}$, $\vec{u}_1^e \in \Re^{L^2 \times 1}$, $\vec{Y}^e \in \Re^{L^2 \times 1}$ and the superscript (e) stands for the even case. The vector $\vec{u}_1^e$ is the vector of water pumpage by all users in period 1, given by $\vec{u}_1^e = [\vec{u}_{q_1}^e \vec{u}_{q_2}^e \vec{u}_{q_3}^e \vec{u}_{q_4}^e]^T$, where $\vec{u}_{q_i}^e$ is the vector of water pumpage in period 1 by all users in quadrant q_i, given by $\vec{u}_{q_i}^e = [u_{(i,1,1),1} u_{(i,2,1),1} \cdots u_{(i,\frac{L-2}{2},\frac{L}{2}),1} u_{(i,\frac{L}{2},\frac{L}{2}),1}]^T$. The vector $\vec{Y}^e$ is the vector of the right hand sides of the FOC, given by $\vec{Y}^e = [\vec{Y}_{q_1}^e \vec{Y}_{q_2}^e \vec{Y}_{q_3}^e \vec{Y}_{q_4}^e]^T$, where

$\vec{Y}^e_{q_i}$ is the right hand side of the FOC corresponding to users in quadrant q_i, given by $\vec{Y}^e_{q_i} = [\lambda \cdots \lambda\eta \cdots \eta\gamma]^T$. The general structure of matrix D^e is given by $D^e = \begin{pmatrix} Q & R & O & S \\ R & Q & P & O \\ O & P & Q & R \\ S & O & R & Q \end{pmatrix}$, where $Q \in \Re^{\frac{L^2}{4} \times \frac{L^2}{4}}$ is the sub matrix of the coefficients of the FOC of user (i, j, k) in quadrant q_i, related to her adjacent users in the same quadrant, $R \in \Re^{\frac{L^2}{4} \times \frac{L^2}{4}}$ is the sub matrix of the coefficients of the FOC of user (i, j, k) in quadrant q_i, related to her adjacent users in the adjacent quadrant q_l, $i \neq l$, for $(i, l) = \{(1, 2), (2, 1), (3, 4), (4, 3)\}$, $O \in \Re^{\frac{L^2}{4} \times \frac{L^2}{4}}$ is the sub matrix of zeros, $P \in \Re^{\frac{L^2}{4} \times \frac{L^2}{4}}$ is the sub matrix of the coefficients of the FOC of user (i, j, k) in quadrant q_2, related to her adjacent users in quadrant q_3 and $S \in \Re^{\frac{L^2}{4} \times \frac{L^2}{4}}$ is the sub matrix of the coefficients of the FOC of user (i, j, k) in quadrant q_1, related to her adjacent users in quadrant q_4. Since quadrants are identical and symmetric, the solution of the FOC implies that users within the same category (corner, edge, internal) pump water equally and symmetrically across the four quadrants in period 1. More specifically, the four corner users pump the same quantity of water in period 1. Edge users having one corner neighbor, one edge neighbor and one internal neighbor pump equally in period 1 and those having two edge neighbors and one internal neighbor pump equally in period 1 as well. Internal users having two edge neighbors and two internal neighbors pump equally in period 1, while those having one edge neighbor and three internal neighbors pump equally in period 1, whereas those having four internal neighbors pump equally in period 1. In the sequel, as will be shown numerically later in our numerical results in the next chapter, we have the following conjecture.

Conjecture 1 (Number of distinct unconstrained solutions of square even grids) *The number of distinct solutions of the FOC for the decentralized problem corresponding to a square even $L \times L$ grid is $\frac{L(L+2)}{8}$, for $L = 4, 6, \cdots, l$ (even).*

For a square odd $L \times L$ grid, by evaluating the FOC in equation (3.39), we have one equation for the central user given by

$$\phi u_{(0,0),1} + \sigma[u_{(0,1),1} + u_{(0,-1),1} + u_{(1,0),1} + u_{(-1,0),1}] = \lambda \tag{3.43}$$

We have, for $j = \pm 1, \cdots, \pm\frac{L-3}{2}$,

$$\phi u_{(j,0),1} + \sigma[u_{(j-1,0),1} + u_{(j+1,0),1} + \Sigma_{(i',j',k')\in S_3'} u_{(i',j',k'),1}] = \lambda \tag{3.44}$$

where S_3' is the set of users (i', j', k') adjacent to central horizontal internal user $(j, 0)$. Similarly, we have, for $k = \pm 1, \cdots, \pm\frac{L-3}{2}$,

$$\phi u_{(0,k),1} + \sigma[u_{(0,k-1),1} + u_{(0,k+1),1} + \Sigma_{(i',j',k')\in S_3'} u_{(i',j',k'),1}] = \lambda \tag{3.45}$$

where S_3' is the set of users (i', j', k') adjacent to central vertical internal user $(0, k)$. We have two equations for $u_{(\pm\frac{L-1}{2},0),1}$ and two equations for $u_{(0,\pm\frac{L-1}{2}),1}$ having, respectively, the following forms

$$\delta u_{(\pm\frac{L-1}{2},0),1} + \epsilon[u_{(\pm\frac{L-3}{2},0),1} + \Sigma_{(i',j',k')\in S_2'} u_{(i',j',k'),1}] = \eta \tag{3.46}$$

$$\delta u_{(0,\pm\frac{L-1}{2}),1} + \epsilon[u_{(0,\pm\frac{L-3}{2}),1} + \Sigma_{(i^*,j^*,k^*)\in S_2^*} u_{(i^*,j^*,k^*),1}] = \eta \tag{3.47}$$

where S_2', S_2^* are, respectively, the sets of users (i', j', k') and (i^*, j^*, k^*) adjacent to the two central horizontal edge users $(\pm\frac{L-1}{2}, 0)$ and to the two central vertical edge users $(0, \pm\frac{L-1}{2})$. Furthermore, we have $(L^2 - 6L + 9)$ equations having the form given in equation (3.40), $4(L - 3)$ equations having the form given in equation (3.41) and four equations having the form given in equation (3.42). Notice that in both cases, even and odd, the total number of equations of the FOC equals L^2. Similar to square even grids, the FOC for a square odd $L \times L$ grid can be written in a general matrix form $D^o\vec{u}_1^o = \vec{Y}^o$, where $D^o \in \Re^{L^2\times L^2}$, $\vec{u}_1^o \in \Re^{L^2\times 1}$, $\vec{Y}^o \in \Re^{L^2\times 1}$ and the superscript (o) denotes the odd case.

The vector $\vec{u}_1^o$ is the vector of water pumpage by all users in period 1, given by $\vec{u}_1^o = [\vec{u}_c \vec{u}_{q_1}^o \vec{u}_{q_2}^o \vec{u}_{q_3}^o \vec{u}_{q_4}^o]^T$, where $\vec{u}_c \in \Re^{(2L-1)\times 1}$ is the vector of water pumpage of the center's user (0,0) and all the horizontal $(\pm j, 0)$ and vertical $(0, \pm k)$ central strips' users, given by $\vec{u}_c = [u_{(0,0),1} u_{(0,1),1} \cdots u_{(0,\frac{L-3}{2}),1} u_{(0,-\frac{L-3}{2}),1} u_{(1,0),1} \cdots u_{(\frac{L-3}{2},0),1} u_{(-\frac{L-3}{2},0),1} u_{(0,\frac{L-1}{2}),1} u_{(0,-\frac{L-1}{2}),1} u_{(\frac{L-1}{2},0),1} u_{(-\frac{L-1}{2},0),1}]^T$. The vector $\vec{u}_{q_i}^o \in \Re^{\frac{(L-1)^2}{4}\times 1}$ is the vector of water pumpage of all users in period 1 of quadrant q_i, given by $\vec{u}_{q_i}^o = [u_{(i,1,1)} u_{(i,2,1)} \cdots u_{(i,\frac{L-3}{2},\frac{L-3}{2})} u_{(i,\frac{L-1}{2},\frac{L-1}{2})}]^T$. The vector $\vec{Y}^o$ is the vector of the right hand sides of the FOC, given by $\vec{Y}^o = [\vec{Y}_c \vec{Y}_{q_1}^o \vec{Y}_{q_2}^o \vec{Y}_{q_3}^o \vec{Y}_{q_4}^o]^T$, where $\vec{Y}_c \in \Re^{(2L-1)\times 1}$ is the right hand side of the FOC corresponding to the users in vector $\vec{u}_c$, given by $\vec{Y}_c = [\lambda\lambda \cdots \lambda\lambda\lambda \cdots \lambda\lambda\eta\eta\eta\eta]^T$ and $\vec{Y}_{q_i}^o$ is the right hand side of the FOC corresponding to users in quadrant q_i, given by $\vec{Y}_{q_i}^e = [\lambda \cdots \lambda\eta \cdots \eta\gamma]^T$.

The general structure of matrix D^o is given by $D^o = \begin{pmatrix} Q_c & V_1 & V_2 & V_3 & V_4 \\ V_1^T & Q & O & O & O \\ V_2^T & O & Q & O & O \\ V_3^T & O & O & Q & O \\ V_4^T & O & O & O & Q \end{pmatrix}$, where $Q_c \in \Re^{(2L-1)\times(2L-1)}$ is the sub matrix of the coefficients of the FOC of users on the central horizontal and vertical strips with their neighbors in the four grid quadrants, $Q \in \Re^{(\frac{L-1}{2})^2\times(\frac{L-1}{2})^2}$ is the sub matrix of the coefficients of the FOC of user (i, j, k) in quadrant q_i, related to her adjacent users within the same quadrant, $V_i \in \Re^{(\frac{L-1}{2})^2\times(2L-1)}$ is the sub matrix of the coefficients of the FOC of user (i, j, k) in quadrant q_i, related to her adjacent users within the central horizontal and vertical strips in the grid and $V_i^T \in \Re^{(2L-1)\times(\frac{L-1}{2})^2}$ is V_i's transpose and $O \in \Re^{(\frac{L-1}{2})^2\times(\frac{L-1}{2})^2}$ is the sub matrix of zeros. Based on similar symmetries we have in even grids, we have the following conjecture, which is derived from observations on our numerical results as will be shown later in the next chapter.

Conjecture 2 (Number of distinct unconstrained solutions of square odd grids) *The number of distinct solutions of the FOC for the decentralized problem corresponding to a square odd* $L \times L$ *grid is* $\frac{(L+1)(L+3)}{8}$*, for* $L = 3, 5, \cdots, l$ *(odd).*

3.6.2 The Centralized Problem

The problem can be stated as a dynamic program as follows. For $t = 1, 2$, and $i = 1, \cdots, n$,

$$\tilde{\Gamma}^*_t(\vec{u}_t, \vec{x}_t) = \max_{u_{i,t},\cdots,u_{n,t}} \tilde{\Gamma}_t(\vec{u}_t, \vec{x}_t) = \max_{u_{i,t},\cdots,u_{n,t}} \{\{\sum_{i=1}^{n} g_{i,t}(u_{i,t}, x_{i,t})\} + \beta_{i,t}\tilde{\Gamma}^*_{t+1}(\vec{u}_{t+1}, \vec{x}_{t+1})\} \tag{3.48}$$

$$s.t. \quad (3.35) \quad and \quad (3.38)$$

where $\tilde{\Gamma}_t(\vec{u}_t, \vec{x}_t)$ is the joint profit-to-go function from period t until the end of the horizon. All of the other conventions and notations of the decentralized problem are retained. Since $\tilde{\Gamma}_t(\vec{u}_t, \vec{x}_t)$ is a positive linear combination of individual discounted profit-to-go functions in the decentralized problem, we immediately have the following.

Corollary 3.14 (Myopic optimality, Positivity, Continuity, Concavity)

(i) The myopically optimal water usage in period t is to deplete all stock ($g^*_{i,t}(u_{i,t}, x_{i,t}) = g_{i,t}(x_{i,t}, x_{i,t})$).

(ii) For a given $\vec{x}_1$, $\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1)$ is strictly increasing at $u_{i,1} = 0$ if $\rho_1 a_1 \geq \beta(\rho_2 a_2 + c_2 w_1)$ for all i.

(iii) For a given $\vec{x}_1$, $\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1)$ is continuous and jointly concave in $\vec{u}_1$ if and only if $c_2 \leq \rho_2 b_2$ and the roots of the function $\lambda^5 + \gamma_0\lambda^4 + \gamma_1\lambda^3 + \gamma_2\lambda^2 + \gamma_3\lambda + \gamma_4 = 0$ are all non-positive, $i = 1, \cdots, n_i$, where n_i is the number of internal users on the grid. The function parameters are as given in Proposition 3.15 (iv).

The results above imply that the centralized problem also reduces to an equivalent single period concave quadratic optimization problem subject to the constraint set $0 < u_{i,1} \leq 1$ for all i. Constructing the FOC of the centralized problem results in the system $\tilde{A}\vec{u}^*_1 = \tilde{W}$, where $\tilde{A}_{n\times n}$ does not have a compact form amenable to obtain structural properties for the solution. However, since users are identical, and by referring to the centralized problem results of the strip configuration, we can determine the number of non-zero elements (coefficients of decision variables) in each row of matrix $\tilde{A}$. More specifically, each row related to a corner user, which has two neighbors, contains 5 non-zero elements under the user's neighbors decision variables and those of their neighbors. For an edge user having three neighbors, each row of the matrix corresponding to that edge user contains 9 non-zero elements under the user's neighbors decision variables and those of their neighbors. For internal users, in which each user has four neighbors, each row contains 13 non-zero elements corresponding to the user's neighbors decision variables and those of their neighbors and those of their neighbors' neighbors. Because of its complicated structure, its large size and the lack of having a sequential indexing of users from 1 to n on the grid, we are not able to derive and write a clear structure of matrix $\tilde{A}$. However, due to the symmetry we have across users and across the grid quadrants, and based on the results we have before for strip and ring configurations' centralized problems, we find that the unconstrained solution of the centralized problem with identical users is unique, symmetric across user and independent of the hydrological properties of the aquifer, α. Furthermore, the unconstrained solution is the optimal for the centralized problem for certain cost and revenue parameter values. We state this result as a conjecture and it will be supported in our numerical results in the next chapter.

Conjecture 3 (Uniqueness and optimality of the global maximizer) *The optimal unconstrained solution of an $n-$identical users grid's centralized problem is unique, independent of the transmissivity coefficient; α, and is equal to those of the strip and ring configurations.*

3.7 A Model with a Salvage Value Function

So far, we have considered the scenario where all water stock is depleted by the end of the problem's time horizon. In this section, we extend our original model by allowing users to partially consume their available water stocks at the beginning of period two for irrigation purposes and to salvage their remaining stocks for other purposes according to a quadratic salvage value function. Salvage value functions are frequently used to overcome some undesirable and unrepresentative behavior at the end of the time horizon. By using salvage value functions, we can modify that end-of-horizon behavior such that the distant future for reasonably long time horizons has little effect on the actions in the preceding periods. Thus, the addition of an appropriate salvage function may be viewed as a proxy for the impact of extending the problem horizon. We discuss this variant of the model below. We limit this extension to the the strip and ring configurations where we investigate the changes that might happen to their main analytical results we have before for their respective decentralized and centralized problems. However, we will not consider the multi (double)-layer ring and the grid configurations in this variant of the original model and we leave them as future extensions. Generally, as will be shown below, we observe that the fundamental results hold under certain conditions for this variant of the model under the strip and ring configurations.

In the analysis below, we keep all the notations and conventions we have before for the strip and ring configurations. However, we relax the condition on $\Gamma^*_{i,3}(.,.)$ which assumes that $\Gamma^*_{i,3}(\vec{u}_3, \vec{x}_3) \equiv 0$ for all $\vec{x}_3$, $\vec{u}_3$ and for $i = 1, \cdots, n$. By relaxing such a condition, we incorporate into our model the possibility of salvaging some of the water stocks available at the beginning of period two for all users. We assume that it is not necessary for the available stock of water at the beginning of period 2, $x_{i,2}$, to be completely consumed in irrigation of crops. More specifically, part of $x_{i,2}$ which represents the pumpage quantity in period 2, $u_{i,2}$, is used to satisfy irrigation demands while the remaining part, $(x_{i,2} - u_{i,2})$, is salvaged by satisfying another source of demand. We assume a quadratic salvage value function for the unused water quantity in period 2, $(x_{i,2} - u_{i,2})$, for $i = 1, \cdots, n$, given by

$$sv_{i,2}(u_{i,2}, x_{i,2}) = \pi_{i,1}(x_{i,2} - u_{i,2}) - 0.5\pi_{i,2}(x_{i,2} - u_{i,2})^2 \tag{3.49}$$

where $\pi_{i,1}$ and $\pi_{i,2}$ are assumed to be positive, $i = 1, \cdots, n$. The advantage of the quadratic salvage function over a linear one is the ease to allocate a partial finite quantity of the water stock available in period 2; $(x_{i,2} - u_{i,2})$, to be salvaged through selling it out to different demand outlet other than the irrigation one. However, with a linear salvage function with a steep positive linear slope, we allocate all of the available stock in period to be salvaged and nothing would remain for satisfying irrigation demands. Therefore, having a quadratic salvage value in period 2 will allow us to study the behavior of users in water usage more realistically by overcoming the undesirable behavior of them at the end of the time horizon. Now, we define $\tilde{g}_{i,2}(u_{i,2}, x_{i,2}) = g_{i,2}(u_{i,2}, x_{i,2}) + sv_{i,2}(u_{i,2}, x_{i,2})$, as the sum of the profit realized from water usage in period 2; $u_{i,2}$, and that realized from salvaging the remaining water stock in period 2; $(x_{i,2} - u_{i,2})$, where $g_{i,2}(u_{i,2}, x_{i,2})$ is as given in Eqn (3.1). To find the optimal water pumpage quantity in period 2, $u^*_{i,2}$, we optimize $\tilde{g}_{i,2}(u_{i,2}, x_{i,2})$ with respect to $u_{i,2}$. More specifically, we solve for the unconstrained solution of $\tilde{g}_{i,2}(u_{i,2}, x_{i,2})$ and determine its feasibility conditions. The unconstrained solution is found by solving the FOC of $\tilde{g}_{i,2}(u_{i,2}, x_{i,2})$. From solving

$$\frac{\partial \tilde{g}_{i,2}(.,.)}{\partial u_{i,2}} = \rho_{i,2}a_{i,2} - c_{i,2}(x_{i,0} - x_{i,2}) - (\rho_{i,2}b_{i,2} + c_{i,2})u_{i,2} - \pi_{i,1} + \pi_{i,2}(x_{i,2} - u_{i,2}) = 0 \tag{3.50}$$

we get the unconstrained unique solution given by

$$u^{**}_{i,2} = \frac{\rho_{i,2}a_{i,2} + (c_{i,2} + \pi_{i,2})x_{i,2} - c_{i,2}x_{i,0} - \pi_{i,1}}{\rho_{i,2}b_{i,2} + c_{i,2} + \pi_{i,2}} \tag{3.51}$$

To guarantee the feasibility of Eqn (3.51), the following condition should hold

$$c_{i,2}x_{i,1} + \pi_{i,1} - (c_{i,2} + \pi_{i,2})x_{i,2} < \rho_{i,2}a_{i,2} < c_{i,2}x_{i,1} + \pi_{i,1} + \rho_{i,2}b_{i,2}x_{i,2} \tag{3.52}$$

By Lemma 3.1 (ii), we know that $g_{i,1}(u_{i,1}, x_{i,1})$ is continuous concave in $u_{i,1}$, for $i = 1, \cdots, n$. Also, we notice that $\partial^2 \tilde{g}_{i,2}(.,.)/\partial(u_{i,2})^2 = -(\rho_{i,2}b_{i,2} + c_{i,2} + \pi_{i,2}) <$

0, implying that $\tilde{g}_{i,2}(u_{i,2}, x_{i,2})$ is continuous and concave in $u_{i,2}$, $i = 1, \cdots, n$. Therefore, $g_{i,1}(u_{i,1}, x_{i,1})$ and $\tilde{g}_{i,2}(u_{i,2}, x_{i,2})$ are continuous and concave in their respective decision variables, for all i. To avoid repetition of writing similar results we have before in the original model of strip and ring configurations, we only present the changes that might appear on the respective results under this new model variant in the decentralized and centralized problems.

Below, we start with the decentralized problem of the strip configuration with salvage value function.

3.7.1 Strip Configuration: The Decentralized Problem-Revisited

For $t = 1, 2$, and $i = 1, \cdots, n$, the decentralized problem of user i is stated as

$$\Gamma^*_{i,1}(\vec{u}_1, \vec{x}_1) = \max_{u_{i,1}} \Gamma_{i,1}(\vec{u}_1, \vec{x}_1) = \max_{u_{i,1}} [g_{i,1}(u_{i,1}, x_{i,1}) + \beta \tilde{g}_{i,2}(u^{**}_{i,2}, x_{i,2})] \qquad (3.53)$$

$$s.t. \quad (3.4) \ \ and \ \ (3.5)$$

The second period's unique solution given above in Eqn (3.51) is a function of the water stocks available at the beginning of period two; $x_{i,2}$, which is, in turn, a function of the first period's decision variables; $u_{i,1}$. Hence, $\tilde{g}_{i,2}(u^{**}_{i,2}, x_{i,2})$ will be a function of $u_{i,1}$. In the sequel, the objective function $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$ will be a function of $\vec{u}_1$ since both $g_{i,1}(u_{i,1}, x_{i,1})$ and $\tilde{g}_{i,2}(u^{**}_{i,2}, x_{i,2})$ are functions of $\vec{u}_1$. The decentralized problem in Eqn (3.53) and Eqn (3.7.1) reduces to a single-period optimization problem. The following result presents some structural properties of $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$.

Proposition 3.17 (Concavity) *The function* $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$ *is continuous and jointly concave in* $\vec{u}_1$ *if and only if*

$2c_{i,2}(c_{i,2}+\pi_{i,2})(\rho_{i,2}b_{i,2}+c_{i,2}+\pi_{i,2})-(\rho_{i,2}b_{i,2})^2\pi_{i,2}-(c_{i,2}+\pi_{i,2})^2(\rho_{i,2}b_{i,2}+c_{i,2}) \leq 0$, $i = 1, \cdots, n$.

Proof *See Appendix.*

Proposition 3.2 (Existence of Nash equilibrium) of the original model holds for this model since the properties of the decentralized problem satisfy the conditions of existence of Nash equilibrium. Furthermore, Proposition 3.3 (Uniqueness of the global maximizer and optimality) of the original model holds as well but with the following new modifications of the parameters

$$\gamma_i = \begin{cases} \frac{\beta(1-\alpha)^2}{y^2}z - (\rho_{i,1}b_{i,1} + c_{i,1}), & i = 1, n \\ \frac{\beta(1-2\alpha)^2}{y^2}z - (\rho_{i,1}b_{i,1} + c_{i,1}), & \text{o.w.} \end{cases},$$

$$\sigma_i = \begin{cases} \frac{\beta\alpha(1-\alpha)}{y^2}z, & i = 1 \\ \frac{\beta\alpha(1-2\alpha)}{y^2}z, & \text{o.w.} \end{cases},$$

$$\epsilon_i = \begin{cases} \frac{\beta\alpha(1-\alpha)}{y^2}z, & i = n-1 \\ \frac{\beta\alpha(1-2\alpha)}{y^2}z, & \text{o.w.} \end{cases} \quad \text{and}$$

$$\lambda_i = \begin{cases} \frac{\beta(1-\alpha)}{y^2}[v_0 y + v_1] - \rho_{i,1}a_{i,1}, & i = 1, n \\ \frac{\beta(1-2\alpha)}{y^2}[v_0 y + v_1] - \rho_{i,1}a_{i,1}, & \text{o.w.} \end{cases}$$

where, for $i = 1, \cdots, n$, $y = \rho_{i,2}b_{i,2} + c_{i,2} + \pi_{i,2}$,

$z = (c_{i,2} + \pi_{i,2})[2c_{i,2}y - (c_{i,2} + \pi_{i,2})(\rho_{i,2}b_{i,2} + c_{i,2})] - (\rho_{i,2}b_{i,2})^2\pi_{i,2}$,

$v_0 = (2c_{i,2} + \pi_{i,2})\rho_{i,2}a_{i,2} + 2c_{i,2}(c_{i,2} + \pi_{i,2})w_1 + c_{i,2}(\pi_{i,2}x_{i,0} - \pi_{i,1})$ and

$v_1 = \rho_{i,2}b_{i,2}[\pi_{i,1}(\rho_{i,2}b_{i,2} + c_{i,2}) - \rho_{i,2}b_{i,2}\pi_{i,2}(x_{i,0} + w_1) + \rho_{i,2}a_{i,2}\pi_{i,2} - c_{i,2}\pi_{i,2}x_{i,0}] - (\rho_{i,2}b_{i,2} + c_{i,2})(c_{i,2} + \pi_{i,2})[\rho_{i,2}a_{i,2} + \pi_{i,2}x_{i,0} + (c_{i,2} + \pi_{i,2})w_1 - \pi_{i,1}]$.

Corollary 3.2 holds with

$\gamma = \frac{\beta(1-\alpha)^2}{y^2}z - (\rho_1 b_1 + c_1)$, $\epsilon = \frac{\beta(1-2\alpha)^2}{y^2}z - (\rho_1 b_1 + c_1)$, $\omega = \frac{\beta\alpha(1-\alpha)}{y^2}z$, $\sigma = \frac{\beta\alpha(1-2\alpha)}{y^2}z$, $\eta = \frac{\beta(1-\alpha)}{y^2}[v_0 y + v_1] - \rho_1 a_1$, $\lambda = \frac{\beta(1-2\alpha)}{y^2}[v_0 y + v_1] - \rho_1 a_1$,

$y = \rho_2 b_2 + c_2 + \pi_2$,

$z = (c_2 + \pi_2)[2c_2 y - (c_2 + \pi_2)(\rho_2 b_2 + c_2)] - (\rho_2 b_2)^2 \pi_2$,

$v_0 = (2c_2 + \pi_2)\rho_2 a_2 + 2c_2(c_2 + \pi_2)w_1 + c_2(\pi_2 x_0 - \pi_1)$ and

$v_1 = \rho_2 b_2[\pi_1(\rho_2 b_2 + c_2) - \rho_2 b_2 \pi_2(x_0 + w_1) + \rho_2 a_2 \pi_2 - c_2 \pi_2 x_0] - (\rho_2 b_2 + c_2)(c_2 + \pi_2)[\rho_2 a_2 + \pi_2 x_0 + (c_2 + \pi_2)w_1 - \pi_1]$.

3.7.2 Strip Configuration: The Centralized Problem-Revisited

The centralized problem under this setting has the same form of that given in Eqn (3.12) subject to the constraints in Eqns (3.4) and (3.7). Similar to the decentralized problem, if the conditions stated in Proposition 3.17 hold, then the objective function of the centralized problem, $\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1)$, is strictly increasing in $u_{i,1}$ at $u_{i,1} = 0$, continuous and jointly concave in $\vec{u}_1$, for all i.

Fundamentally, Proposition 3.4 of the original model holds under this setting as well. However, the derivation of the corresponding coefficients $e_{(i,i)}$ and the right hand sides, θ_i, for $i = 1, \cdots, n$, is very messy, and, hence, we skip writing their formulae for strips with non-identical users. Nevertheless, for identical users, the elements of matrix $\tilde{A}$ and the right hand side $\tilde{W}$ in Section 3.2.2 become as follows

$\phi_1 = -(\rho_1 b_1 + c_1) + \beta(1 - 2\alpha + 2\alpha^2)\frac{\tilde{z}}{y}$, $\phi_2 = \beta(2\alpha - 3\alpha^2)\frac{\tilde{z}}{y}$, $\phi_3 = \beta\alpha^2\frac{\tilde{z}}{y}$, $\omega_1 = -(\rho_1 b_1 + c_1) + \beta(1 - 4\alpha + 6\alpha^2)\frac{\tilde{z}}{y}$, $\omega_2 = \phi_1 - \omega_1 = 2\beta\alpha(1 - 2\alpha)\frac{\tilde{z}}{y}$ and $\theta = \beta\frac{\tilde{y}}{y} - \rho_1 a_1$, where $y = \rho_2 b_2 + c_2 + \pi_2$, $\tilde{z} = y[(c_2 + 1)(c_2 + \pi_2)y + (\rho_2 b_2 + c_2)(c_2 + \pi_2)^2] - (\rho_2 b_2)^2 \pi_2$ and $\tilde{y} = (c_2 + \pi_2)(\rho_2 a_2 - c_2 w_1) + (\rho_2 a_2 + \pi_2 x_1 - c_2 w_1 - \pi_1)[1 + (c_2 + \pi_2)(\rho_2 b_2 + c_2)] - \rho_2 b_2 \pi_1 - \frac{\pi_2}{y}[\rho_2 a_2 - \rho_2 b_2 - c_2(x_1 + w_1) - \pi_1]$.

The following result gives the solution of the centralized problem for identical users which is equivalent to Corollary 3.4 and, hence, its proof is omitted.

Corollary 3.15 (Uniqueness of the global maximizer and optimality for identical users)

(i) Suppose that users are identical. Then, $\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1)$ *is unique and given by*

$$u_{i,1}^{**} = u^{**} = [\beta\tilde{y} - \rho_1 a_1 y]/[\beta\tilde{z} - (\rho_1 b_1 + c_1)y], \ \forall i$$

(ii) If $0 \leq u_{i,1}^{**} \leq x_1$*, for all* i*, then the optimal solution for the centralized problem is given by* u^{**} *above.*

Corollary 3.5 of the original model (no coordination of the two solutions) holds in this setting as well.

3.7.3 Ring Configuration: The Decentralized Problem-Revisited

Similar to the strip's decentralized problem, $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$ is strictly increasing in $u_{i,1}$ at $u_{i,1} = 0$, continuous and jointly concave in $\vec{u}_1$ provided that their condition in Proposition 3.17 are satisfied for all i. In accordance with that, the decentralized problem under this setting possesses at least one Nash equilibrium as well and, hence, Proposition 3.6 holds under this setting. Moreover, Proposition 3.7 holds with the following modifications

$\epsilon_i = \frac{\beta(1-2\alpha)^2}{y^2} z - (\rho_{i,1} b_{i,1} + c_{i,1})$, $\sigma_i = \frac{\beta\alpha(1-2\alpha)}{y^2} z$, $\lambda_i = \frac{\beta(1-2\alpha)}{y^2}[v_0 y + v_1] - \rho_{i,1} a_{i,1}$ and y, z, v_0 and v_1 are as defined before in the decentralized problem of the strip configuration is Section 3.7.1. Furthermore, Corollary 3.6 holds in this setting as well and gives the corresponding unique Nash equilibrium for identical users in the ring configuration.

3.7.4 Ring Configuration: The Centralized Problem-Revisited

Again, similar to the strip's centralized problem, if the conditions on $\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1)$ hold, then, the objective function is strictly increasing in $u_{i,1}$ at $u_{i,1} = 0$, continuous and jointly concave in $\vec{u}_1$, for all i. Fundamentally, Proposition 3.8 of the original model holds under this setting. However, the derivation of the respective coefficients $e_{(i,i)}$ and the right hand sides ϕ_i, for $i = 1, \cdots, n$, is very messy, and, hence, we skip writing their formulae similar to the non-identical setting in the strip configuration.

Nevertheless, in the identical case, we observe that the elements of matrix $\tilde{B}$ in Section 3.3.2 are ϕ_i, $i = 1, 2, 3$, ω_i, $i = 1, 2$ and θ are as defined above in matrix $\tilde{A}$ for the centralized problem of the strip configuration in Section 3.7.2. The optimal solution of the centralized problem corresponding to the ring configuration gives the same solution given in Corollary 3.8. Moreover, Corollary 3.9 (no coordination between the two solutions) holds under this setting as well.

In the next chapter, we will give some numerical examples for identical users in strip and ring configurations to illustrate the water usage behavior of users when they can salvage some of their water stocks in period two.

3.8 Summary

In this chapter, we investigated the decentralized and centralized groundwater management problems for a groundwater aquifer with multi non-identical users with time-variant parameters over a planning horizon of two periods. Five different geometrical user configurations were considered for analysis; namely, strip, ring, double-layer ring, multi-layer ring and grid configurations. Existence of Nash equilibria was established for each strategic-form game corresponding to the decentralized problems under each geometric configuration. Except the grid configuration, unique Nash equilibrium was characterized under the decentralized management scheme for each of the remaining configurations. More compact Nash equilibria solutions could be derived for the case of identical users under the first four configurations. Also, unique optimal equilibrium usage was determined under the centralized management scheme for each configuration except the grid one.

When users are identical, unique, aquifer transmissivity-independent, geometric configuration-independent and equal across users water usage quantity solutions were obtained for all configurations under the centralized management scheme. The analysis of the management problems for the grid configuration was more tedious compared to that in the other configurations. However, we could bring forth some conjectures related to the number of distinct unconstrained solutions of the decentralized problem for square grids. Also, we could bring forth a conjecture about the centralized solution for square grids. These conjectures are supported by our numerical study in the next chapter. To overcome the undesirable effect of users behavior at the end of the time horizon, we added an appropriate quadratic salvage function to the original model and revisited the analysis of the management problems for strip and ring configurations. The addition of the salvage value function could be considered as a proxy of having an extended time horizon. We found that all the results we have established before in the strip and ring configurations' management problems still hold under the salvage value function model but under different conditions. It is worth pointing out that the analysis under the salvage value function model was more burdensome than that in the original model. In all configurations, it was shown that coordination between the centralized solution and the decentralized one could not be achieved via a single pricing mechanism.

Chapter 4

Numerical Results for Groundwater Usage Model

In the previous chapter, we gave the analytical results of the decentralized and centralized problems for a group of distinct geometric configurations of users overlaying and sharing a common groundwater aquifer region. In this chapter, we present the results of some hypothetical numerical examples illustrating and comparing between the decentralized and centralized solutions under different parameters settings. In all of the following examples, for the purposes of interpretation and comparison of the results, all users are taken to be identical. More specifically, in strip and ring configurations, we consider the results of identical users in order to be able to interpret and compare them with the results in Saak and Peterson [52]. However, for the other configurations, our analytical results are already derived for identical users.

In Section 4.1, we discuss the impact of the number of users on the optimal water usage and expected profits in both decentralized and centralized problems of the strip configuration. We consider three different parameter settings. Namely, the time-invariant setting of all revenue-cost parameters, the time-variant setting with varying unit price of the crops and the time-variant setting with varying yield function parameter. Section 4.2 presents the corresponding numerical results of the impact of the number of users on the optimal water usage and expected profits for both problems under the same three settings in Section 4.1, but for the ring configuration. In Section 4.3, we study the effect of the lateral transmissivity coefficient on the optimal water usage and expected profits in both decentralized and centralized problems for a fixed number of users in the strip and ring configurations. Section 4.4 summarizes the numerical results of the impact of the number of users on the optimal solutions and the expected profits in both problems of both strip and ring configurations but for two different discount rates under time-invariant settings.

In Section 4.5, we study the effect of allowing users to salvage part of their water stock in the second period on the optimal water usage and the realized profits. In Section 4.6, we present the numerical results corresponding to a double-layer ring configuration and to one multi-layer ring configuration. Section 4.7 includes the optimal water usage and the expected profit values of the decentralized and centralized problems corresponding to different square grids. We discuss these numerical results and verify Conjectures (1)-(3) in Section 3.6.

4.1 Impact of Number of Users in a Strip

In this section, we now present some numerical examples to illustrate the impact of the number of users the optimal usage quantities and the discounted profits. All users are taken on as identical with parameters $\beta = 1$, $w_{i,0} = w_{i,1} = 0$ and $x_{i,0} = x_{i,1} = 1$. For comparison with the results of Saak and Peterson [52], we assume that α is perceived by all users to be a random variable uniformly distributed over $[0, 0.5]$. We provide numerical examples for time-invariant and time-variant settings. We investigate the impact of the number of users on optimal water usage and expected profits in the strip configuration. We consider three different settings in this example. Namely, the first one is time-invariant in which, we set $\rho_{i,t} = 1$, $a_{i,t} = 10$, $b_{i,t} = 5$ and $c_{i,t} = 2$ for all i and t. The second setting is time-variant in which, we set $\rho_{i,1} = 1.05$, $\rho_{i,2} = 1$, $a_{i,t} = 10$, $b_{i,t} = 5$ and $c_{i,t} = 2$ for all i and t. The last setting is also time variant in which, we set $\rho_{i,t} = 1$,

$a_{i,1} = 10.5$, $a_{i,2} = 10$, $b_{i,t} = 5$ and $c_{i,t} = 2$ for all i and t. Table 4.1 summarizes the water usage per user in period 1 accompanied with the total discounted profits in the centralized ($\tilde{\Gamma}_1^*$) and decentralized ($\Gamma_{i,1}^*$) problems realized over the two-period horizon. The centralized solution is found from Corollary 3.4. More specifically, for time-invariant setting, we find that $u_{i,1}^* = 0.5$, $i = 1, ..., n$, and the corresponding discounted profit is 7.83. For n users, the total discounted profit attained by the social planner is $7.83n$. In the second setting, we have $u_{i,1}^* = 0.5366$, $i = 1, ..., n$, the discounted profit per user is 7.98 and the total discounted profit of the social planner is $7.98n$. Likewise, in the last setting, we have $u_{i,1}^* = 0.55$, $i = 1, ..., n$, the discounted profit per user is 8.01 and total discounted profit attained by the social planner is $8.01n$. We observe that with higher crop's unit price in period 1 (setting 2), users pump more in period 1 and realize more total profits in the centralized problem compared to time-invariant price (setting 1). However, as they pump more under this setting, their total profits in the decentralized problem deteriorate with respect to (w.r.t.) the time-invariant setting. In setting 3, we observe that users pump more in period 1 and realize more total profit compared to the time-invariant setting in both centralized and decentralized problems.

Table 4.2 presents the total usage per user over the two-period planning horizon ($u_{i,1}^* + u_{i,2}^*$) of the decentralized problem with time-invariant setting. In this table, TP_t denotes the total usage in period t where ($TP_t = \sum_{i=1}^n u_{i,t}^*$) and $R_t\%$ denotes the percentage of the average usage, ($R_t\%$=$(TP_t/n)\times 100\%$). The optimal water usage is symmetric around the mid-point of the strip but not monotone w.r.t. the user location. This numerically validates Saak and Peterson's conjecture as noted in Section 3.2.1. We make two observations. *(i)* Non-extreme users pump more than the extreme ones in period 1, while the opposite is true in period 2. *(ii)* The total water usage may exceed the initial stock levels for some users. Table 4.3 tabulates the total discounted profit per user in the decentralized problem. We note that the profits are consistent with the total water usage; that is, highest profits are obtained by the second to extreme users. Likewise, profits are also symmetric around the midpoints and non-monotone. However, the least profits are not realized by the extreme users, which may be attributed

n	1	2	3	4	5	6	7	8	9	10
Setting 1	$\rho_{i,t} = 1$, $a_{i,t} = 10$, $b_{i,t} = 5$, $c_{i,t} = 2$									
$u^*_{1,1}$	.5	.6757	.6631	.6635	.6635	.6635	.6635	.6635	.6635	.6635
$u^*_{2,1}$	—	.6757	.8961	.8890	.8892	.8892	.8892	.8892	.8892	.8892
$u^*_{3,1}$	—	—	.6631	.8890	.8819	.8821	.8821	.8821	.8821	.8821
$u^*_{4,1}$	—	—	—	.6635	.8892	.8821	.8824	.8824	.8824	.8824
$u^*_{5,1}$	—	—	—	—	.6635	.8892	.8821	.8824	.8824	.8824
$u^*_{6,1}$	—	—	—	—	—	.6635	.8892	.8821	.8824	.8824
$u^*_{7,1}$	—	—	—	—	—	—	.6635	.8892	.8821	.8824
$u^*_{8,1}$	—	—	—	—	—	—	—	.6635	.8892	.8821
$u^*_{9,1}$	—	—	—	—	—	—	—	—	.6635	.8892
$u^*_{10,1}$	—	—	—	—	—	—	—	—	—	.6635
$\tilde{\Gamma}^*_1(\vec{u}_1, \vec{x}_1)$	7.83	15.66	23.49	31.32	39.15	46.98	54.81	62.64	70.47	78.30
$\sum_{i=1}^n \Gamma^*_{i,1}(\vec{u}_1, \vec{x}_1)$	7.83	15.20	22.26	29.28	36.08	43.12	50.14	57.16	64.18	71.20
Setting 2	$\rho_{i,1} = 1.05$, $\rho_{i,2} = 1$, $a_{i,t} = 10$, $b_{i,t} = 5$, $c_{i,t} = 2$									
$u^*_{1,1}$	.5366	.7105	.6985	.6988	.6988	.6988	.6988	.6988	.6988	.6988
$u^*_{2,1}$	—	.7105	.9274	.9206	.9208	.9208	.9208	.9208	.9208	.9208
$u^*_{3,1}$	—	—	.6985	.9206	.9124	.9141	.9141	.9141	.9141	.9141
$u^*_{4,1}$	—	—	—	.6988	.9208	.9141	.9143	.9143	.9143	.9143
$u^*_{5,1}$	—	—	—	—	.6988	.9208	.9141	.9143	.9143	.9143
$u^*_{6,1}$	—	—	—	—	—	.6988	.9208	.9141	.9143	.9143
$u^*_{7,1}$	—	—	—	—	—	—	.6988	.9208	.9141	.9143
$u^*_{8,1}$	—	—	—	—	—	—	—	.6988	.9208	.9141
$u^*_{9,1}$	—	—	—	—	—	—	—	—	.6988	.9208
$u^*_{10,1}$	—	—	—	—	—	—	—	—	—	.6988
$\tilde{\Gamma}^*_1(\vec{u}_1, \vec{x}_1)$	7.98	15.96	23.94	31.92	39.90	47.88	55.86	63.84	71.82	79.80
$\sum_{i=1}^n \Gamma^*_{i,1}(\vec{u}_1, \vec{x}_1)$	7.98	15.64	21.91	28.12	35.07	42.06	49.02	55.98	62.94	69.90
Setting 3	$\rho_{i,t} = 1$, $a_{i,1} = 10.5$, $a_{i,2} = 10$, $b_{i,t} = 5$, $c_{i,t} = 2$									
$u^*_{1,1}$	.55	.7297	.7168	.7172	.7172	.7172	.7172	.7172	.7172	.7172
$u^*_{2,1}$	—	.7297	.9552	.9480	.9482	.9482	.9482	.9482	.9482	.9482
$u^*_{3,1}$	—	—	.7168	.9480	.9407	.9410	.9410	.9410	.9410	.9410
$u^*_{4,1}$	—	—	—	.7172	.9482	.9410	.9412	.9412	.9412	.9412
$u^*_{5,1}$	—	—	—	—	.7172	.9482	.9410	.9412	.9412	.9412
$u^*_{6,1}$	—	—	—	—	—	.7172	.9482	.9410	.9412	.9412
$u^*_{7,1}$	—	—	—	—	—	—	.7172	.9482	.9410	.9412
$u^*_{8,1}$	—	—	—	—	—	—	—	.7172	.9482	.9410
$u^*_{9,1}$	—	—	—	—	—	—	—	—	.7172	.9482
$u^*_{10,1}$	—	—	—	—	—	—	—	—	—	.7172
$\tilde{\Gamma}^*_1(\vec{u}_1, \vec{x}_1)$	8.01	16.02	24.03	32.04	40.05	48.06	56.07	64.08	72.09	80.10
$\sum_{i=1}^n \Gamma^*_{i,1}(\vec{u}_1, \vec{x}_1)$	8.01	15.7	23	30	37.49	44.73	51.96	59.21	66.46	73.71

Table 4.1: Equilibrium usage in period 1 for $n-$identical users on a strip

to the non-linear nature of the profit function. It is worth noting that, under the time-variant settings, users exhibit the same behavior in pumpage and in profit realization, and, hence, we skip giving their results.

n	1	2	3	4	5	6	7	8	9	10
$u^*_{1,1}+u^*_{1,2}$	1	1	.9418	.9436	.9436	.9436	.9436	.9436	.9436	.9436
$u^*_{2,1}+u^*_{2,2}$	—	1	1.1165	1.0564	1.0564	1.0582	1.0582	1.0582	1.0582	1.0582
$u^*_{3,1}+u^*_{3,2}$	—	—	.9418	1.0564	.9928	.9982	.9981	.9981	.9981	.9981
$u^*_{4,1}+u^*_{4,2}$	—	—	—	.9436	1.0564	.9982	1.0002	1.0002	1.0002	1.0002
$u^*_{5,1}+u^*_{5,2}$	—	—	—	—	.9436	1.0582	.9982	1.0002	1.0002	1.0002
$u^*_{6,1}+u^*_{6,2}$	—	—	—	—	—	.9436	1.0582	.9981	1.0002	1.0002
$u^*_{7,1}+u^*_{7,2}$	—	—	—	—	—	—	.9436	1.0582	.9981	1.0002
$u^*_{8,1}+u^*_{8,2}$	—	—	—	—	—	—	—	.9436	1.0582	.9981
$u^*_{9,1}+u^*_{9,2}$	—	—	—	—	—	—	—	—	.9436	1.0582
$u^*_{10,1}+u^*_{10,2}$	—	—	—	—	—	—	—	—	—	.9436
TP_1	.5	1.351	2.223	3.105	3.994	4.870	5.752	6.634	7.516	8.400
$R_1\%$	50	67.57	74.08	77.62	79.75	81.16	82.17	82.93	83.52	83.99

Table 4.2: Total equilibrium usage and total profits for $n-$identical users on a strip: time-invariant setting

4.2 Impact of Number of Users in a Ring

We now consider the ring configuration with the same parameter setting in Section 4.1. Table 4.4 summarizes the corresponding numerical results. We observe that users pump more and realize more profits under the time-variant settings compared to the time-invariant one. The corresponding centralized solutions are found from Corollary 3.8, which are the same as those found in Corollary 3.4 above in the strip configuration. Figure (4.1a) depicts the values of $R_1\%$ versus the number of users n for strip and ring configurations for the data tabulated in Table 4.1 and Table 4.4 corresponding to the time-invariant setting. We observe that $R_1\%$ increases concavely in the number of users. This implies that users become more greedy as more users share the resource, however the tendency to pump more water diminishes. As expected, for both configurations, the maximum discounted profits are attained in the centralized problem. However, for $n \geq 3$,

n	1	2	3	4	5	6	7	8	9	10
$\Gamma^*_{1,1}(\vec{u}_1,\vec{x}_1)$	7.83	7.60	7.21	7.21	7.12	7.12	7.12	7.12	7.12	7.12
$\Gamma^*_{2,1}(\vec{u}_1,\vec{x}_1)$	—	7.60	7.83	7.42	7.42	7.44	7.44	7.44	7.44	7.44
$\Gamma^*_{3,1}(\vec{u}_1,\vec{x}_1)$	—	—	7.21	7.42	7.00	7.00	7.00	7.00	7.00	7.00
$\Gamma^*_{4,1}(\vec{u}_1,\vec{x}_1)$	—	—	—	7.22	7.42	7.00	7.02	7.02	7.02	7.02
$\Gamma^*_{5,1}(\vec{u}_1,\vec{x}_1)$	—	—	—	—	7.12	7.44	7.00	7.02	7.02	7.02
$\Gamma^*_{6,1}(\vec{u}_1,\vec{x}_1)$	—	—	—	—	—	7.12	7.44	7.00	7.02	7.02
$\Gamma^*_{7,1}(\vec{u}_1,\vec{x}_1)$	—	—	—	—	—	—	7.12	7.44	7.00	7.02
$\Gamma^*_{8,1}(\vec{u}_1,\vec{x}_1)$	—	—	—	—	—	—	—	7.12	7.44	7.00
$\Gamma^*_{9,1}(\vec{u}_1,\vec{x}_1)$	—	—	—	—	—	—	—	—	7.12	7.44
$\Gamma^*_{10,1}(\vec{u}_1,\vec{x}_1)$	—	—	—	—	—	—	—	—	—	7.12
$\sum_{i=1}^{n}\Gamma^*_{i,1}(\vec{u}_1,\vec{x}_1)$	7.83	15.20	22.26	29.28	36.08	43.12	50.14	57.16	64.18	71.20

Table 4.3: Profits per user in the decentralized problem for $n-$identical users on a strip: time-invariant setting

the strip configuration yields more discounted profits than the ring configuration in the decentralized problem. This occurs because, users in the strip configuration exhibit an oscillating greedy behavior of pumpage in period 1 where they pump more water than they do in the ring configuration. Again, it is worth noting that users show the same behavior in their $R_1\%$ under the time-variant settings and, hence, their corresponding figures are not given.

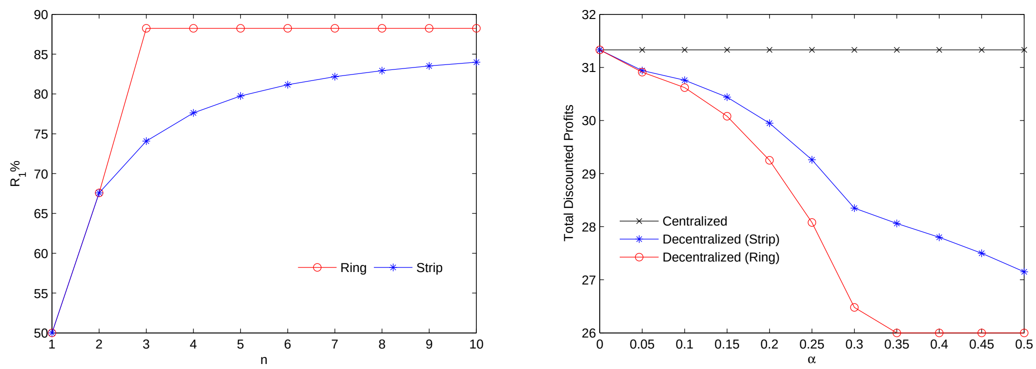
4.3 Impact of Lateral Transmissivity in a Strip and a Ring

In this example, we examine the effect of $\alpha \in [0, 0.5]$ on the total decentralized discounted profits in both configurations for $n = 4$ identical users. Here, we assume that users have perfect information about the soil transmissivity and treat α as a deterministic parameter. We set $\rho_{i,t} = 1$, $a_{i,t} = 10$, $b_{i,t} = 5$ and $c_{i,t} = 2$ for all i and t, (*i.e.*, the time-invariant setting). Tables 4.5 and 4.6 summarize the results for the strip and ring configurations, respectively. In both tables, $\triangle P\% = [(\tilde{\Gamma}^*_1 - \Gamma^*_{i,1})]/\tilde{\Gamma}^*_1 \times 100\%$ stands for the percentage rate of decrease

n	$u^*_{i,1}$	$u^*_{i,2}$	$(u^*_{i,1}+u^*_{i,2})$	TP_1	$R_1\%$	$\tilde{\Gamma}^*_1(\vec{u}_1,\vec{x}_1)$	$\sum_{i=1}^{n}\Gamma^*_{i,1}(\vec{u}_1,\vec{x}_1)$
Setting 1	$\rho_{i,t}=1$, $a_{i,t}=10$, $b_{i,t}=5$, $c_{i,t}=2$						
1	.5	.5	1	.5	50	7.83	7.83
2	.6757	.3243	1	1.3514	67.57	15.66	15.20
$n\geq 3$	.8824	.1176	1	$.8824n$	88.24	$7.83n$	$7.032n$
Setting 2	$\rho_{i,1}=1.05$, $\rho_{i,2}=1$, $a_{i,t}=10$, $b_{i,t}=5$, $c_{i,t}=2$						
1	.5366	.4634	1	.5366	53.66	7.98	7.98
2	.7105	.2895	1	1.421	71.05	15.96	15.64
$n\geq 3$	.9143	.0857	1	$.9143n$	91.43	$7.98n$	$7.24n$
Setting 3	$\rho_{i,t}=1$, $a_{i,1}=10.5$, $a_{i,2}=10$, $b_{i,t}=5$, $c_{i,t}=2$						
1	.55	.45	1	.55	55	8.01	8.01
2	.7297	.2703	1	1.4594	72.97	16.02	15.7
$n\geq 3$	.9411	.0589	1	$.9411n$	94.11	$8.01n$	$7.25n$

Table 4.4: Equilibrium usage in periods 1 and 2 and total profits for $n-$identical users on a ring

in discounted profit of the decentralized problem relative to that in the centralized problem. In the strip configuration, the unconstrained solution for $\alpha\in[0.35,0.5]$, resulted in infeasible solutions; $u^{**}_{2,1}=u^{**}_{3,1}>1$ and $u^{**}_{1,1}=u^{**}_{4,1}<1$. Hence, we obtained the constrained solution numerically, $u^*_{2,1}=u^*_{3,1}=1$, $(i.e.\ \delta^*_2=\delta^*_3>0)$ and $u^*_{1,1}=u^*_{4,1}<1$, $(i.e.\ \delta^*_1=\delta^*_4=0)$. Similarly, the unconstrained solutions are suboptimal for the ring configuration for $\alpha\in[0.35,0.5]$. The optimal solution obtained numerically results in all users are depleting their total available stock of water in period 1, $u^*_{i,1}=1$ and $\delta^*_i=0$, for $i=1,2,3,4$. We note that in both configurations, as α increases, users experience more effects of hydrologic dynamics and become more greedy tending to use more water in period 1. Figure (4.1b) depicts the total discounted profits w.r.t. α in both configurations. As observed from the figure, the total discounted profits are non-increasing in α regardless of the configuration. However, the rate of decrease, $\triangle P\%$, in the strip configuration is always lower than that in the ring for $\alpha\in[0,0.50]$. It is important to note that in both configurations, the maximum discounted profit is attained in the centralized setting where the realized total discounted profit is 31.33. However, both centralized and decentralized problems achieve the same value of total discounted profits when there is no lateral flow between users ($i.e.$,



(a) $R_1\%$ vs. n: Decentralized problem

(b) Identical case ($n = 4$): Total discounted profit vs. α

Figure 4.1: ($R_1\%$ vs. n: Decentralized problem), (Total discounted profits vs. α): time-invariant setting

when $\alpha = 0$), as expected.

α	$(u^*_{1,1}, u^*_{1,2})$	$(u^*_{2,1}, u^*_{2,2})$	$(u^*_{3,1}, u^*_{3,2})$	$(u^*_{4,1}, u^*_{4,2})$	$\tilde{\Gamma}^*_1(\vec{u}_1, \vec{x}_1)$	$\sum_{i=1}^{n} \Gamma^*_{i,1}(\vec{u}_1, \vec{x}_1)$	$\triangle P\%$
0	(.5,.5)	(.5,.5)	(.5,.5)	(.5,.5)	31.33	31.33	0
.05	(.5325,.4658)	(.5675,.4343)	(.5675,.4343)	(.5325,.4658)	31.33	30.94	1.25
.10	(.5649,.4276)	(.6402,.3673)	(.6402,.3673)	(.5649,.4276)	31.33	30.76	1.82
.15	(.5972,.3846)	(.7185,.2997)	(.7185,.2997)	(.5972,.3846)	31.33	30.44	2.84
.20	(.6295,.3359)	(.8025,.2321)	(.8025,.2321)	(.6295,.3359)	31.33	29.95	4.4
.25	(.6616,.2807)	(.8925,.1652)	(.8925,.1652)	(.6616,.2807)	31.33	29.26	6.6
.30	(.6939,.2177)	(.9885,.0999)	(.9885,.0999)	(.6939,.2177)	31.33	28.35	9.51
.35	(.7339,.1729)	(1,.0931)	(1,.0931)	(.7339,.1729)	31.33	28.06	10.4
.40	(.7772,.1337)	(1,.0891)	(1,.0891)	(.7772,.1337)	31.33	27.80	11.27
.45	(.8230,.0974)	(1,.0797)	(1,.0797)	(.8230,.0974)	31.33	27.50	12.22
.50	(.8709,.0646)	(1,.0646)	(1,.0646)	(.8709,.0646)	31.33	27.15	13.34

Table 4.5: Total discounted profit vs. α: strip configuration

α	$(u^*_{1,1}, u^*_{1,2})$	$(u^*_{2,1}, u^*_{2,2})$	$(u^*_{3,1}, u^*_{3,2})$	$(u^*_{4,1}, u^*_{4,2})$	$\tilde{\Gamma}^*_1(\vec{u}_1, \vec{x}_1)$	$\sum_{i=1}^{n} \Gamma^*_{i,1}(\vec{u}_1, \vec{x}_1)$	$\triangle P\%$
0	(.5,.5)	(.5,.5)	(.5,.5)	(.5,.5)	31.33	31.33	0
.05	(.5670,.4330)	(.5670,.4330)	(.5670,.4330)	(.5670,.4330)	31.33	30.91	1.34
.10	(.6383,.3617)	(.6383,.3617)	(.6383,.3617)	(.6383,.3617)	31.33	30.62	2.17
.15	(.7143,.2857)	(.7143,.2857)	(.7143,.2857)	(.7143,.2857)	31.33	30.08	4
.20	(.7955,.2045)	(.7955,.2045)	(.7955,.2045)	(.7955,.2045)	31.33	29.25	6.64
.25	(.8824,.1176)	(.8824,.1176)	(.8824,.1176)	(.8824,.1176)	31.33	28.08	10.4
.30	(.9756,.0244)	(.9756,.0244)	(.9756,.0244)	(.9756,.0244)	31.33	26.48	15.5
.35	(1,0)	(1,0)	(1,0)	(1,0)	31.33	26	17
.40	(1,0)	(1,0)	(1,0)	(1,0)	31.33	26	17
.45	(1,0)	(1,0)	(1,0)	(1,0)	31.33	26	17
.50	(1,0)	(1,0)	(1,0)	(1,0)	31.33	26	17

Table 4.6: Total discounted profit vs. α: ring configuration

4.4 Impact of Discount Rate in a Strip and a Ring

In the following two examples, we investigate the effect of the discount rate β on the optimal solution both centralized and decentralized problems for different numbers of users under time-invariant parameters setting. More specifically, we set $a_{i,t} = 10$, $b_{i,t} = 5$ and $c_{i,t} = 2$ for all i and t. Also, we take $w_{i,1} = 0$ and $x_{i,1} = 1$, for all i. We assume that α is perceived by all users to be a random variable uniformly distributed over $[0, 0.5]$. Notice that the previously-mentioned conditions on parameters are satisfied in this case as well. First, we take $\beta = 0.9$. Under this setting, from Proposition 5, it is easy to show that $u^{**}_{i,1} = u^*_{i,1} = 0.5670$, $i = 1, ..., n$, and the corresponding discounted profit is 7.73. For n users, the total discounted profit attained by the social planner is $7.73n$. Table 4.7 summarizes the results corresponding to the strip configuration.

For the ring configuration, from Proposition 10, the optimal water usage in period 1 is $u^{**}_{i,1} = u^*_{i,1} = 0.5670$, $i = 1, ..., n$, and the associated discounted profit is 7.73, whereas the total discounted profit of the social planner is $7.73n$. From Proposition 9, we find the solution of the decentralized problem. Table 4.8 summarizes the corresponding results of the ring configuration.

n	1	2	3	4	5	6	7	8	9	10
$u^*_{1,1}$	.5670	.7341	.7233	.7236	.7236	.7236	.7236	.7236	.7236	.7236
$u^*_{2,1}$	—	.7341	.9398	.9338	.9340	.9340	.9340	.9340	.9340	.9340
$u^*_{3,1}$	—	—	.7233	.9338	.9278	.9278	.9278	.9278	.9278	.9278
$u^*_{4,1}$	—	—	—	.7236	.9240	.9278	.9282	.9282	.9282	.9282
$u^*_{5,1}$	—	—	—	—	.7236	.9340	.9278	.9282	.9282	.9282
$u^*_{6,1}$	—	—	—	—	—	.7236	.9340	.9278	.9282	.9282
$u^*_{7,1}$	—	—	—	—	—	—	.7236	.9340	.9278	.9282
$u^*_{8,1}$	—	—	—	—	—	—	—	.7236	.9340	.9278
$u^*_{9,1}$	—	—	—	—	—	—	—	—	.7236	.9340
$u^*_{10,1}$	—	—	—	—	—	—	—	—	—	.7236
$\tilde{\Gamma}^*_1(\vec{u}_1, \vec{x}_1)$	7.73	15.46	23.19	30.92	38.65	46.38	54.11	61.84	69.57	77.30
$\Gamma^*_{i,1}(\vec{u}_1, \vec{x}_1)$	7.73	14.95	21.83	28.67	35.50	42.06	49.17	56.08	62.91	67.74

Table 4.7: Equilibrium usage in period 1 for $n-$identical users strip: $\beta = 0.9$

n	$u^*_{i,1}$	$\tilde{\Gamma}^*_1(\vec{u}_1, \vec{x}_1)$	$\Gamma^*_{i,1}(\vec{u}_1, \vec{x}_1)$
1	.5670	7.73	7.73
2	.7341	15.46	14.95
$n \geq 3$	.9281	$7.73n$	$6.83n$

Table 4.8: Equilibrium usage in period 1 for $n-$identical users ring: $\beta = 0.9$

Now, we take $\beta = 0.75$. From Proposition 5, it is easy to show that $u^{**}_{i,1} = u^*_{i,1} = 0.6757,\ i = 1, ..., n$, and the corresponding discounted profit is 7.60. For n users, the total discounted profit attained by the social planner is $7.60n$. Table 4.9 summarizes the results corresponding to the strip configuration.

Regarding the ring configuration, from Proposition 10, the optimal water usage in period 1 is $u^{**}_{i,1} = u^*_{i,1} = 0.6757,\ i = 1, ..., n$, and the associated discounted profit is 7.60, whereas the total discounted profit of the social planner is $7.60n$. From Proposition 9, we find the solution of the decentralized problem. Table 4.10 summarizes the corresponding results of the ring configuration.

n	1	2	3	4	5	6	7	8	9	10
$u^*_{1,1}$	.6757	.8273	.8192	.8194	.8914	.8914	.8914	.8914	.8914	.8914
$u^*_{2,1}$	—	.8273	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$u^*_{3,1}$	—	—	.8912	1.0000	.9998	.9998	.9999	.9999	.9999	.9999
$u^*_{4,1}$	—	—	—	.8914	1.000	.9998	1.000	1.000	1.000	1.000
$u^*_{5,1}$	—	—	—	—	.8914	1.000	.9999	1.000	1.000	1.000
$u^*_{6,1}$	—	—	—	—	—	.8914	1.000	.9999	.9999	1.000
$u^*_{7,1}$	—	—	—	—	—	—	.8914	1.000	.9999	1.000
$u^*_{8,1}$	—	—	—	—	—	—	—	.8914	1.000	.9999
$u^*_{9,1}$	—	—	—	—	—	—	—	—	.8914	1.000
$u^*_{10,1}$	—	—	—	—	—	—	—	—	—	.8914
$\tilde{\Gamma}^*_1(\vec{u}_1, \vec{x}_1)$	7.60	15.20	22.80	30.40	38.00	45.60	53.20	60.80	68.40	76.00
$\Gamma^*_{i,1}(\vec{u}_1, \vec{x}_1)$	7.60	14.29	21.01	27.52	34.02	40.52	47.02	53.52	60.02	66.52

Table 4.9: Equilibrium usage in period 1 for $n-$identical users strip: $\beta = 0.75$

n	$u^*_{i,1}$	$\tilde{\Gamma}^*_1(\vec{u}_1, \vec{x}_1)$	$\Gamma^*_{i,1}(\vec{u}_1, \vec{x}_1)$
1	.6757	7.60	7.60
2	.8273	15.20	14.29
$n \geq 3$	1.000	$7.60n$	$6.50n$

Table 4.10: Equilibrium usage in period 1 for $n-$identical users ring: $\beta = 0.75$

4.5 Impact of Salvage Value in a Strip and a Ring

We next present a numerical example to illustrate the effect of allowing users to salvage some of their water stock in period 2. W.l.o.g., we consider a system with $n = 6$ users and solve for the centralized and decentralized problems in both strip and ring configurations. All users are taken as identical with parameters $\beta = 1$, $\alpha = 0.25$, $\rho_t = 1$, $a_t = 10$, $b_t = 5$, $c_t = 1$, for $t = 1, 2$, $x_1 = 1$ and $w_1 = 0$. We consider two settings; one with salvage model having $\pi_1 = 8$ and $\pi_2 = 5$, and the other without salvage (original model) having $\pi_t = 0$, for $t = 1, 2$. It can be shown that the selected parameters guarantee the concavity

i	1	2	3	4	5	6
Setting 1	*Salvage Model* ($d_1 = 8$, $d_2 = 5$)					
$u^*_{i,1}$	.5697	.7449	.7424	.7424	.7449	.5697
$x_{i,2}$	.3865	.2995	.2570	.2570	.2995	.3865
$u^*_{i,2}$	.2371	.2000	.1816	.1816	.2000	.2371
$(x_{i,2} - u^*_{i,1})$	.1494	.0995	.0754	.0754	.0995	.1494
$\Gamma^*_{i,1}(\vec{u}_1, \vec{x}_1)$	7.347	7.403	7.064	7.064	7.403	7.347
Setting 2	$Non - Salvage\ Model$ ($d_1 = d_2 = 0$)					
$u^*_{i,1}$	.6706	.9567	.9362	.9362	.9567	.6706
$x_{i,2}$	.2579	.1120	.0587	.0587	.1120	.2579
$u^*_{i,2}$	.2579	.1120	.0587	.0587	.1120	.2579
$(x_{i,2} - u^*_{i,1})$	0	0	0	0	0	0
$\Gamma^*_{i,1}(\vec{u}_1, \vec{x}_1)$	7.545	7.804	7.254	7.254	7.804	7.545

Table 4.11: Optimal solution of the decentralized problem-strip configuration: salvage and non-salvage models

of the objective functions of both problems under both settings as well as they satisfy the conditions in Proposition 3.17. Table 4.11 summarizes the optimal solution of the decentralized problem accompanied with the realized discounted profits for the strip configuration under both settings. The total profit realized from water usage by all users is 43.63 in the salvage model while its is 45.206 in the non-salvage model. In the ring configuration, Table 4.12 summarizes the corresponding results of the decentralized problem under both settings.

The optimal solution of the centralized problem in both configurations under both settings is found from Proposition 3.5.1. More specifically, in the salvage model setting, for both strip and ring configurations, it is found that the centralized solution is given by $u^*_{i,1} = 0.0377$, $x_{i,2} = 0.9623$, $u^*_{i,2} = 0.4838$ and $(x_{i,2} - u^*_{i,2}) = 0.4785$, for $i = 1, \cdots, 6$. Each user achieves a total discounted profit of 7.393, and, hence, the total profit realized from water usage by all users is 44.36. In the non-salvage model setting, the centralized solution of strip and ring configurations is $u^*_{i,1} = 0.5$, $x_{i,2} = 0.5$, $u^*_{i,2} = 0.5$ and $(x_{i,2} - u^*_{i,2}) = 0$, for $i = 1, \cdots, 6$. The total discounted profit of each user is 8.25, and, hence, the total profit realized from water usage by all users is 49.5. We observe that, in

Soln.	$u_{i,1}^*$	$x_{i,2}$	$u_{i,2}^*$	$(x_{i,2} - u_{i,1}^*)$	$\Gamma_{i,1}^*(\vec{u}_1, \vec{x}_1)$	$\sum_{i=1}^{n=6} \Gamma_{i,1}^*(\vec{u}_1, \vec{x}_1)$
Setting 1	*Salvage Model* $(d_1 = 8, d_2 = 5)$					
$1 \le i \le 6$	.7425	.2575	.1818	.0757	7.07	42.42
Setting 2	$Non - Salvage\ Model$ $(d_1 = d_2 = 0)$					
$1 \le i \le 6$	.9375	.0625	.0625	0	7.293	43.76

Table 4.12: Optimal solution of the decentralized problem-ring configuration: salvage and non-salvage models

both configurations under both settings, the centralized solution dominates the decentralized one by realizing more profits from water usage. Also, in strip configuration, the water usage fluctuates from ends toward the midpoints of the strip. Under the salvage model setting, in both configurations and in both problems, users allocate part of their available water stocks in the second period to satisfy demands other than the irrigation ones through selling it out according to the given salvage value function. In the sequel, under this setting, the policy makers (users and social planner) have more flexibility in allocating their water stock in the second period among two different sources of water demand.

4.6 Numerical Study of Multi-Layer Rings

In this section, we present two numerical examples; one for a double-layer ring configuration and the other for a five-layer ring configuration.

4.6.1 Numerical Results for a Double-Layer Ring

The double-layer ring configuration is considered a special case of the general multi-layer ring configuration with $m = 2$. In this setting, all users within the same layer are taken to be identical. Specifically, the parameters of the profit function in Eqn (3.18) are selected such that the conditions in Proposition 3.9 and Corollary 3.11 pertinent to the positivity, continuity and concavity of

$\Gamma_{(i,k),1}(\vec{u}_1, \vec{x}_1)$ and $\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1)$ in $u_{(i,k),1}$ are satisfied. To this end, relevant parameters are chosen as follows: $\rho_1 = 1$, $\rho_2 = 2$, $a_1 = 15$, $a_2 = 15$, $b_1 = 8$, $b_2 = 6$, $c_1 = 10$ and $c_2 = 8$. Also, we set $\beta = 1$ and we assume that there is no aquifer recharge; ($i.e.\ w_{k,1} = 0,\ k = 1, 2$), while we assume different initial water stocks across the two layers with values given by $x_{1,1} = 1$ and $x_{2,1} = 2$.

Let us index the inside layer in Figure 3.4 by $k = 1$ and the outside one by $k = 2$. We assume the same transmissivity coefficient among users with the same layer. Hence, in this numerical example, we set $\alpha_k = 0.1$, $k = 1, 2$ while we set the transmissivity coefficient among the two layers to $\alpha = 0.25$. The decentralized problem solution is found from Proposition 3.11 while the centralized problem solution is found from Proposition 3.12. Table 4.13 summarizes the optimal solution of the centralized and decentralized problems accompanied with their total discounted profits realized from water usage by all users over the two-period planning horizon.

	Decentralized Solution			Centralized Solution		
layer-k	$u^*_{(i,k),1}$	$u^*_{(i,k),2}$	$\Gamma^*_{(i,k),1}(\vec{u}_1, \vec{x}_1)$	$u^*_{(i,k),1}$	$u^*_{(i,k),2}$	$\tilde{\Gamma}^*_{(i,k),1}(\vec{u}_1, \vec{x}_1)$
1	0.8164	0.3617	6.71	0.9155	0.2686	6.58
2	1.104	0.7179	0.25	1.179	0.6369	0.54
Total	$1.9204n$	$1.0796n$	$6.96n$	$2.0945n$	$0.9055n$	$7.12n$

Table 4.13: Decentralized and centralized solutions of a double-ring configuration

The tabulated results in Table 4.13 are given for any number of users n in the system. The water usage of users in this setting resembles to a great extent the water usage of two non-identical users in a strip configuration. We observe that users in the inside layer ($k = 1$) would prefer the decentralized management scheme as they realize more profits compared to those realized in the centralized management scheme. The opposite is true for users in the outside layer ($k = 2$). However, when we consider the total discounted profits realized by all users in both layers, we observe that the centralized solution dominates the decentralized one by realizing more total discounted profits over the entire time horizon. Consequently, even it is harmful to users in layer one, the centralized management scheme results in more social welfare compared to the decentralized one.

4.6.2 Numerical Results for a Multi-Layer Ring

In this setting, we consider a multi-layer ring configuration with five layers ($m = 5$), each contains a number of n identical users. Similar to the double layer configuration, the parameters of the profit function in Eqn (3.18) are selected such that the conditions in Proposition 3.5.1 and Corollary 5.1 pertinent to the positivity, continuity and concavity of $\Gamma_{(i,k),1}(\vec{u}_1, \vec{x}_1)$ and $\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1)$ in $u_{(i,k),1}$ are satisfied. The profit function-related parameters in the previous section are adopted for this configuration as well. More specifically, we choose $\rho_1 = 1$, $\rho_2 = 2$, $a_1 = 15$, $a_2 = 15$, $b_1 = 8$, $b_2 = 6$, $c_1 = 10$ and $c_2 = 8$. Also, we set $\beta = 1$ and we assume that there is no aquifer recharge; ($i.e.$ $w_{k,1} = 0,\ k = 1, \cdots, m$) similar to the double-layer ring configuration. We assume different initial water stocks across the five layers with values given by $x_{1,1} = 0.2$, $x_{2,1} = 0.4$, $x_{3,1} = 0.6$, $x_{4,1} = 0.8$ and $x_{5,1} = 1$. The five layers are indexed from inside to outside, where the inmost layer to the center is layer with $k = 1$ while the outmost layer from the center is layer with $k = 5$. Furthermore, we take the transmissivity coefficient among users within the same layer as $\alpha_k = 0.25,\ \forall k$. The set of transmissivity coefficients between users across the five layers are given as follows: $\alpha_{(1,2)} = 0.1$, $\alpha_{(2,3)} = 0.2$, $\alpha_{(3,4)} = 0.1$ and $\alpha_{(4,5)} = 0.2$ with the conventions $\alpha_{(0,1)} = \alpha_{(5,6)} = 0$.

The five-layer ring configuration is equivalent to the strip configuration with five non-identical users, where each user represents one layer. In the sequel, the solutions of the decentralized and centralized problems corresponding to this configuration are found from Proposition 3.3 and Proposition 3.4, respectively. Table 4.14 summarizes the optimal solution of the centralized and decentralized problems accompanied with their total discounted profits realized from water usage by all users over the two-period planning horizon. The tabulated results in Table 4.13 are given for any number of users n in the system.

We notice that users in all layers pump more water in period 1 under the centralized management scheme. This behavior in pumpage could be attributed

	Decentralized Solution			Centralized Solution		
layer-k	$u^*_{(i,k),1}$	$u^*_{(i,k),2}$	$\Gamma^*_{(i,k),1}(\vec{u}_1, \vec{x}_1)$	$u^*_{(i,k),1}$	$u^*_{(i,k),2}$	$\tilde{\Gamma}^*_{(i,k),1}(\vec{u}_1, \vec{x}_1)$
1	0.2	0	2.64	0.2	0	2.64
2	0.4	0	4.56	0.4	0.012	4.64
3	0.6	0.0032	5.78	0.5401	0.0547	5.76
4	0.7684	0.0669	6.44	0.672	0.1454	6.50
5	0.7761	0.1854	6.52	0.7509	0.2249	6.54
Total	$2.7445n$	$0.2555n$	$25.94n$	$2.563n$	$0.437n$	$26.08n$

Table 4.14: Decentralized and centralized solutions of a multi-ring configuration

to the tendency of users within one layer to be more greedy and pump water in period to get benefit from the lateral flows of water from adjacent users in neighboring layers. However, this greedy behavior benefits no body in the system. More specifically, except users in layer one and two who are indifferent to both management schemes, users in layers four and five would prefer to pump water under the supervision of the social planner to realize more profits. Users in layer three would prefer the decentralized scheme and this is intuitively expected from them since they are in the middle layer of the configuration where they are expected to have the largest lateral flows from neighboring layers. Also, we observe that users occupying the first three layers completely consume their initial water stocks in period one when they pump decennially. Since the initial water stocks increase as we move away from the center across the layers, upon water pumpage, water is naturally expected to laterally flow from outside layers to the inside ones as more water stocks are available in the outside layers. Because of that, users in the first three layers completely pump their initial water stocks in period 1 in order to get benefit from those lateral flows. More specifically, under the decentralized scheme, we notice that users in layer three got some water from layer four lateral flows and, similarly, under the centralized scheme, users in layer two got some water from layer three lateral flows. Overall, the centralized solution dominates the decentralized one by realizing more total discounted profits from all users in the system.

4.7 Numerical Study of Square Grids

We present some numerical examples for even and odd square grids. In all examples, users are taken as identical with parameters $\rho = 1$, $\beta = 1$, $\alpha = 0.1$, $a = 10$, b= 5 and $c = 2$. We solve the decentralized and centralized problems for grids having $L = 3$ and $K = 2$, ($i.e.$ $n = 6$), identical users and for square grids with $L = 3, \cdots, 8$. For the sake of simplicity for presenting the numerical results, we adopt a different notation for the optimal pumpage quantity in period 1 based on the category of users on the grid (corner, edge, internal). Specifically, instead of using $u^*_{(i,j,k),1}$ to indicate the decision variable of user (j, k) in quadrant i in period 1, we consider the notation $u^*_{v,1} \in S_m$ to indicate, in period 1, the decision variable of a user having index v in category m, $m = 1, 2, 3$ and $v = 1, \cdots, s$, where s is the maximum number of users in category m. Table 4.15 summarizes the optimal solution of the centralized and decentralized problems accompanied with their total discounted profits realized from water usage by all users over the two-period planning horizon. The results are tabulated for $((3 \times 2), (3 \times 3)$ and $(4 \times 4))$ identical grids and presented based on the new adopted notation, where again $\triangle P\% = [(\tilde{\Gamma}^*_1 - \Gamma^*_{i,1})]/\tilde{\Gamma}^*_1 \times 100\%$ stands for the percentage rate of decrease in discounted profit of the decentralized problem relative to that in the centralized problem.

Case	(L, K)	n	$u^*_{v,1}$	$\tilde{\Gamma}^*_1(\vec{u}_1, \vec{x}_1)$	$u^*_{v,1} \in S_1$	$u^*_{v,1} \in S_2$	$u^*_{v,1} \in S_3$	$\Gamma^*_1(\vec{u}_1, \vec{x}_1)$	$\triangle P\%$
1	(3,2)	6	0.5	47	.6362	.7181	0	42	10.6
2	(3,3)	9	0.5	71	.6341	.7161	.8026	69	2.8
3	(4,4)	16	0.5	125	.6342	.7141	.7990	120	4

Table 4.15: Numerical results for some grid structures

From Conjecture 3, the optimal equilibrium pumpage quantities of water in period 1 of the centralized problem is $u^*_{v,1} = 0.5, \forall\ v = 1, ..., n$. In the decentralized problem, we observe that users exhibit a symmetrical behavior in their water pumpage along the horizontal and vertical strips of the grid. Also, we notice that corner users pump less than edge users who, in turn, pump less than internal users. The centralized solution always dominates the decentralized one by realizing more total discounted profits over the two-period horizon. Observe that, the number of different solutions of the (3×3) and (4×4) grids' decentralized problems is 3, which agrees Conjectures 1 and 2. For a square odd grid of size (5×5), Table 4.16 summarizes the optimal solution of the decentralized problem.

0.6342	**0.7142**	**0.7122**	0.7142	0.6342
0.7142	**0.7989**	**0.7972**	0.7989	0.7142
0.7122	0.7972	**0.7953**	0.7972	0.7122
0.7142	0.7989	0.7972	0.7989	0.7142
0.6342	0.7142	0.7122	0.7142	0.6342

Table 4.16: Equilibrium pumpage in period 1 for a (5×5) grid structure: the decentralized problem

0.7177	**0.8299**	**0.8271**	0.8271	0.8299	0.7177
0.8299	**0.9363**	**0.9340**	0.9340	0.9363	0.8299
0.8271	0.9340	**0.9316**	0.9316	0.9340	0.8271
0.8271	0.9340	0.9316	0.9316	0.9340	0.8271
0.8299	0.9363	0.9340	0.9340	0.9363	0.8299
0.7177	0.8299	0.8271	0.8271	0.8299	0.7177

Table 4.17: Equilibrium pumpage in period 1 for a (6×6) grid structure: the decentralized problem

We observe that the number of different unconstrained solutions of the decentralized problem (denoted in bold) is 6 which agrees with Conjecture 2. Also, we observe the symmetry in water pumpage between users within each category (corner, edge, internal) and across the four quadrants of the grid. Based on Conjecture 3, the optimal solution of the centralized problem is given by $u^*_{v,1} = 0.5, \forall\, v = 1, ..., n$, with total discounted profit of 196. The total discounted profit of the decentralized problem is 187 yielding a profit relative difference, $\triangle P\%$, of 4.6%. Table 4.17 summarizes the optimal solution of the decentralized problem for a (6×6) grid and Table 4.18 summarizes those for a (7×7) grid.

0.7177	**0.8299**	**0.8271**	**0.8272**	0.8271	0.8299	0.7177
0.8299	**0.9363**	**0.9340**	**0.9341**	0.9340	0.9363	0.8299
0.8271	0.9340	**0.9317**	**0.9318**	0.9317	0.9340	0.8271
0.8272	0.9341	0.9318	**0.9319**	0.9318	0.9341	0.8272
0.8271	0.9340	0.9317	0.9318	0.9317	0.9340	0.8271
0.8299	0.9363	0.9340	0.9341	0.9340	0.9363	0.8299
0.7177	0.8299	0.8271	0.8272	0.8271	0.8299	0.7177

Table 4.18: Equilibrium pumpage in period 1 for a (7×7) grid structure: the decentralized problem

We observe that the numbers of different solutions (denoted in bold) in both grids are, respectively, 6 and 10, which coincide with Conjectures 1 and 2. Both structures keep the symmetry in water pumpage between users within each category (corner, edge, internal) and across the quadrants of the grid. Table 4.19 summarizes the corresponding results for a (8×8) grid.

0.7177	**0.8299**	**0.8271**	**0.8272**	0.8272	0.8271	0.8299	0.7177
0.8299	**0.9363**	**0.9340**	**0.9341**	0.9341	0.9340	0.9363	0.8299
0.8271	0.9340	**0.9317**	**0.9318**	0.9318	0.9317	0.9340	0.8271
0.8272	0.9341	0.9318	**0.9319**	0.9319	0.9318	0.9341	0.8272
0.8272	0.9341	0.9318	0.9319	0.9319	0.9318	0.9341	0.8272
0.8271	0.9340	0.9317	0.9318	0.9318	0.9317	0.9340	0.8271
0.8299	0.9363	0.9340	0.9341	0.9341	0.9340	0.9363	0.8299
0.7177	0.8299	0.8271	0.8272	0.8272	0.8271	0.8299	0.7177

Table 4.19: Equilibrium pumpage in period 1 for a (8×8) grid structure: the decentralized problem

The optimal solution of the centralized problem in all grids; (6×6), (7×7) and (8×8), is the same and is given by $u^*_{v,1} = 0.5, \forall\ v = 1, ..., n$, according to Proposition 3.3. Table 4.20 presents the total profit values of the centralized and decentralized problems of (6×6), (7×7) and (8×8) grids. Also, we observe that increasing the number of users from 25 in the (5×5) grid to 36 in the (6×6) grid has a significant effect on the pumpage quantities of all users in the original

(5×5) grid. More specifically, all users in the system pump more water in period 1 relative to what they pump in the original (5×5) grid because having more users in the system makes them more greedy, and, hence, pump more water. However, the effect of increasing the number of users from 36 in the (6×6) grid to 49 in the (7×7) grid does not have a significant affect on the original pumpage quantities of users in the original (6×6) grid. This is also true when we have more users in the (8×8) grid as tabulated in Table 4.19 below.

Case	(L, K)	n	$u^*_{v,1}$	$\tilde{\Gamma}^*_1(\vec{u}_1, \vec{x}_1)$	*Decent. Soln.*	$\Gamma^*_1(\vec{u}_1, \vec{x}_1)$	$\triangle P\%$
1	(6,6)	36	0.5	282.24	*Table* 4.17	242.27	14.16
2	(7,7)	49	0.5	384.16	*Table* 4.18	329.00	14.36
3	(8,8)	64	0.5	501.76	*Table* 4.19	428.21	14.66

Table 4.20: Total profit values of (6×6), (7×7) and (8×8) grids

4.8 Summary

In this chapter, we provided a numerical study for some hypothetical examples to compare water usage behavior of users under the decentralized and centralized management schemes for an aquifer with the five geometric configurations discussed in Chapter 3. To facilitate the ability of comparing and interpreting the solutions, all users were assumed to be identical in all numerical examples. Both time-variant and time-invariant cost-revenue parameter settings were considered in this study. More specifically, we discussed the impact of the number of users on the optimal water usage and expected profits in both decentralized and centralized problems for strip and ring configurations. We observed that the optimal Nash equilibrium of non-extreme users in strip oscillates around the Nash equilibrium in ring as number of users increases. This implies that as number of users increases, the effect of extreme users and those immediately next to them on strip is diminishing as number of users increases, which results in the convergent of the Nash equilibrium of extreme users to that in ring configuration. Also, we noticed that with high aquifer activity (high value of lateral transmissivity coefficient), users become more greedy and pump more water in the initial period of the horizon. However, this behavior deteriorates the total discounted profits realized when the water system is managed/controlled under the decentralized management scheme.

We also investigated the effect of having a quadratic salvage function in the second period on water usage under both decentralized and centralized settings for both strip and ring configurations. We noticed that, under some parameter settings, users optimally exhaust part of their available stock in satisfying irrigation demands and salvage the remnant through selling it out to satisfy another demand outlet. Two numerical examples; one on double-layer ring configuration and the other on five-layer ring configuration, were presented. The decentralized and centralized solutions were found and compared under each configuration. The optimal solutions in the double-layer configuration were found to be unique while those in the five-layer configuration were found to follow the solution structures in strip configuration with non-identical users. The water management problems of square girds of different sizes were studied numerically. We solved for the optimal water usages in both decentralized and centralized problems. We found that the numbers of distinct unconstrained solutions corresponding to both problems in all grids coincide with the conjectures we have mentioned before in the previous chapter. Total centralized and decentralized profits were computed and compared for each grid.

In all of the numerical examples of this chapter, we observe that the centralized solutions always dominate the decentralized ones, both solved under the same configuration and parameter settings, by achieving more total discounted profits realized from water usage by all users in the system. In the sequel, the maximum profit of water usage from all users could be realized when the water system is managed centrally by the social planner. Recall that in the previous chapter local water authorities (social planner) could not establish the possibility of coordinating the centralized and the decentralized solutions via a single pricing mechanism. That is, local water authorities could not impose a feasible unit pumping cost on users to make their non-cooperative usage behavior the same as when the water system is managed centrally. Having the inability of coordinating the two solution makes the implementation of the centralized solution more challenging and difficult in reality. Therefore, to entice users to behave in a central manner in their water pumpage, other considerations should be taken into account in applying the centralized management scheme by local water authorities (social planner). For instance, some incentive systems or structures should be adopted by local water authorities granted to users to encourage them to apply the centralized solution in their water usage. Depending on which party (water authorities or users) in the water system who possesses the property (water) and access rights to the groundwater aquifer region, such incentive structures could be established to apply the centralized solution in order to get the greatest welfare (profit) from water usage. To implement such a centralized management system, other administrative and usage control costs could be incurred and, hence, they should be taken into consideration as well.

Chapter 5

Centralized and Decentralized Management of Conjunctive Use of Surface and Groundwater

As mentioned in Chapter 2, the previous works on groundwater management and conjunctive use management motivate us to consider a more comprehensive and more realistic model in reality with two non-identical users over a planning horizon of two periods. Specifically, the model incorporates the conjunctive use of ground and surface water in a setting that permits the sharing of groundwater stock in an aquifer possessing a finite transmissivity coefficient. This commonality of groundwater results in a game-theoretic dynamic structure among users who use their own private sources of surface water in conjunction with the common groundwater aquifer in order to satisfy their irrigation demands. Users acquire their private surface water stocks from an external supplier (external reservoir) and keep them at their own local reservoirs to be used conjunctively with groundwater.

In this chapter, we discuss the analysis of the the conjunctive water use model. In Section 5.1, we give a detailed description of the model, the main assumptions and some structural properties of water usage profit function. In Section 5.2, we discuss the analytical solution of the decentralized problem. In Section 5.3, we present the analytical solution of the centralized problem. In the last section of this chapter; Section 5.4, we present some illustrative numerical examples.

5.1 The Model

In this section, we present the assumptions and basic properties of the model to be used in our analysis. We consider two non-identical users, each having her own reservoir of surface water in addition to a common groundwater aquifer shared with her neighbor, as depicted in Figure 5.1.

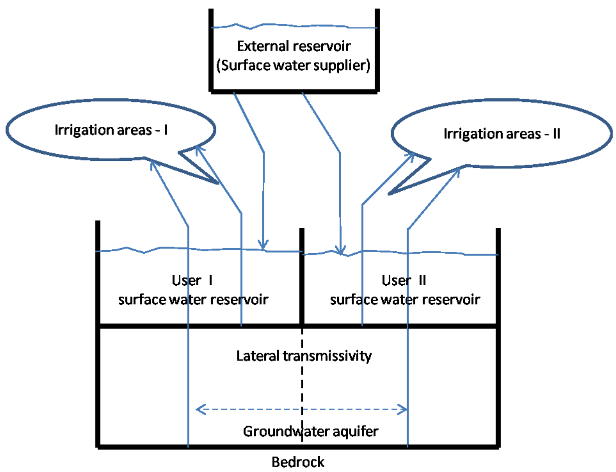


Figure 5.1: Conjunctive surface and groundwater use model

Next, we introduce the notation of this model.

Notation

$x^g_{i,t}$: groundwater stock level at the beginning of period t for user i

$x^g_{i,0}$: initial groundwater stock level at the beginning of the planning horizon for user i; $(x^g_{i,0} = x^g_{i,1}, \forall i)$

$u^g_{i,t}$: groundwater pumpage (and usage) quantity by user i in period t

$u^{g*}_{i,t}$: groundwater optimal pumpage (and usage) quantity by user i in period t

$x^s_{i,t}$: surface water stock level at the beginning of period t for user i

$x^s_{i,0}$: initial surface water stock level at the beginning of the planning horizon for user i; $(x^s_{i,0} = x^s_{i,1}, \forall i)$

$u^s_{i,t}$: surface water usage quantity by user i in period t

$u^{s*}_{i,t}$: surface water optimal usage quantity by user i in period t

$w_{i,1}$: total water usage from surface and groundwater sources by user i in period one; $w_{i,1} = u^g_{i,1} + u^s_{i,1}$

$w^*_{i,1}$: total optimal water usage from surface and groundwater sources by user i in period one; $w^*_{i,1} = u^{g*}_{i,1} + u^{s*}_{i,1}$

α: groundwater aquifer's transmissivity (lateral flow) coefficient; $\alpha \in [0, 0.5]$

$\beta_{i,t}$: discount rate for user i in period t

$a_{i,t}$: output price of the crop when the crop production quantity is zero for user i in period t

$b_{i,t}$: rate of decrease in crop's output price with respect to the crop's production for user i in period t

$c_{i,t}, d_{i,t}$: respectively, linear and quadratic coefficients of groundwater extraction (pumpage) cost function for user i in period t

$h_{i,t}$: cost of holding one unit of surface water for user i

$Q_{i,j}$: lateral flow of groundwater in period one among users i and j; $i \neq j$

$\vec{u}_{i,t}$: surface and groundwater usage vector for user i in period t

$\vec{x}_{i,t}$: surface and groundwater stock vector for user i in period t

$\vec{u}_t$: surface and groundwater usage vector for both users in period t

$\vec{x}_t$: surface and groundwater stock vector for both users in period t

$R_{i,t}(\vec{u}_{i,t}, \vec{x}_{i,t})$: revenue function of water (surface and ground) usage for user i in period t

$R^*_{i,t}(\vec{u}_{i,t}, \vec{x}_{i,t})$: revenue function of optimal water (surface and ground) usage for user i in period t

$R_i(\vec{u}_1, \vec{x}_1)$: total discounted revenue function of water (surface and ground) usage for user i

$C_{i,t}(\vec{u}_{i,t}, \vec{x}_{i,t})$: cost function of water (surface and ground) usage for user i in period t

$C^*_{i,t}(\vec{u}_{i,t}, \vec{x}_{i,t})$: cost function of optimal water (surface and ground) usage for user i in period t

$C_i(\vec{u}_1, \vec{x}_1)$: total discounted cost function of water (surface and ground) usage for user i

$C^g_{i,t}(\vec{u}_{i,t}, \vec{x}_{i,t})$: groundwater extraction (pumpage) cost function for user i in period t

$C^s_{i,t}(\vec{u}_{i,t}, \vec{x}_{i,t})$: surface water average holding cost function for user i in periods t and $t+1$

$f_{i,t}(\vec{u}_{i,t}, \vec{x}_{i,t})$: water (surface and ground) usage profit function for user i in period t

$\Gamma_{i,t}(\vec{u}_t, \vec{x}_t)$: total discounted profit of water (ground and surface) usage for user i in period t

$\Gamma^*_{i,t}(\vec{u}_t, \vec{x}_t)$: maximum total discounted profit of water (ground and surface) usage for user i in period t

$\tilde{R}_1(\vec{u}_1, \vec{x}_1)$: total discounted revenue function for both users' water (surface and ground) usage

$\tilde{C}_1(\vec{u}_1, \vec{x}_1)$: total discounted cost function for both users' water (surface and ground) usage

$\tilde{\Gamma}_t(\vec{u}_t, \vec{x}_t)$: total discounted profit of water (ground and surface) usage for both users in period t

$\tilde{\Gamma}^*_t(\vec{u}_t, \vec{x}_t)$: maximum total discounted profit of water (ground and surface) usage for both users in period t

We assume that possible surface water losses due to evaporation are negligible and that there is no aquifer recharge. At the beginning of period t, user i has access to an underground water stock of $x^g_{i,t}$ and also has access to surface water in her own reservoir with a stock of $x^s_{i,t}$, $i = 1, 2$, $t = 1, 2$. We assume that users have equal initial stock levels of water from both sources. Let $u^s_{i,t}$ denote the amount of surface water released from user i's reservoir (and consumed) by user i in period t. Then, $u^s_{i,t}$ is bounded by $x^s_{i,t}$, which implies that surface water is essentially a private resource within each period, and that in any period, user i can not release more than her stock level of surface water available in her reservoir at the beginning of the same period. We also let $u^g_{i,t}$ denote the amount of groundwater pumped (and consumed) by user i in period t, where $u^g_{i,t}$ is bounded by $x^g_{i,t}$. The groundwater is also a private resource within each period - a user can not access groundwater lying beneath the other user within a given period. Although the groundwater stock in a period for a user is inaccessible by the other, as water levels change locally due to consumption of each user, there will be some lateral flow in the aquifer between users (between the adjacent areas corresponding to users' plots). We assume that lateral flow occurs at the end of a period immediately before the next period begins. The inter-period lateral flow of groundwater is governed by Darcy's Law. In accordance with this law, in period 1, there will be a lateral flow, $Q_{j,i}$, of groundwater from user j to user i given by $Q_{j,i} = -\alpha[(x^g_{i,1} - u^g_{i,1}) - (x^g_{j,1} - u^g_{j,1})] = \alpha(u^g_{i,1} - u^g_{j,1})$ for $i, j = 1, 2,\ i \neq j$, where $\alpha \in [0, 0.5]$ is the finite lateral flow (aquifer transmissivity) coefficient, summarizing the hydrologic dynamics of the groundwater aquifer, and $(x^g_{i,1} - u^g_{i,1}) - (x^g_{j,1} - u^g_{j,1})$ is the hydrologic gradient. The profit functions of users are of quadratic form similar to those in the works of Saak and Peterson [52] and Saleh et al. [54]. Let $\vec{u}_{i,t} = (u^g_{i,t}, u^s_{i,t})$ denote the vector of groundwater and surface water usage by user i in period t and $\vec{x}_{i,t} = (x^g_{i,t}, x^s_{i,t})$ denote the vector of groundwater and surface water stock levels at user i at the beginning of period

t. The profit derived from water usage by user i within period t is given by

$$f_{i,t}(\vec{u}_{i,t}, \vec{x}_{i,t}) = R_{i,t}(\vec{u}_{i,t}, \vec{x}_{i,t}) - C_{i,t}(\vec{u}_{i,t}, \vec{x}_{i,t}) \tag{5.1}$$

where $R_{i,t}(\vec{u}_{i,t}, \vec{x}_{i,t})$ is the periodic water usage revenue function of user i and $C_{i,t}(\vec{u}_{i,t}, \vec{x}_{i,t})$ is the periodic water usage cost function of user i. In particular, $R_{i,t}(\vec{u}_{i,t}, \vec{x}_{i,t})$ is a quadratic function of the revenue derived from the yield of irrigated crops and is given by

$$R_{i,t}(\vec{u}_{i,t}, \vec{x}_{i,t}) = \rho_{i,t} a_{i,t}(u^g_{i,t} + u^s_{i,t}) - 0.5\rho_{i,t} b_{i,t}(u^g_{i,t} + u^s_{i,t})^2 \tag{5.2}$$

where $\rho_{i,t}$ is the periodic price per unit of an irrigated crop and $a_{i,t}, b_{i,t}$ are the periodic crop yield function parameters. We assume that the unit acquisition costs of water from both sources are equal and proportional to water usage quantity, and, hence, they are implicitly included in the first (linear) term of $R_{i,t}(\vec{u}_{i,t}, \vec{x}_{i,t})$. On the other hand, $C_{i,t}(\vec{u}_{i,t}, \vec{x}_{i,t})$ is the sum of two cost components. Namely, the first component is the periodic groundwater pumpage cost given by

$$C^g_{i,t}(u^g_{i,t}, x^g_{i,t}) = \int_0^{u^g_{i,t}} [c_{i,t}(x^g_{i,0} - x^g_{i,t}) + 2d_{i,t} z] dz = c_{i,t}(x^g_{i,0} - x^g_{i,t}) u^g_{i,t} + d_{i,t}(u^g_{i,t})^2,$$

which is a quadratic function of both the difference between the base stock level; $x^g_{i,0}$, and the stock level of groundwater at the beginning of period t as well as the pumpage (and usage) quantity of groundwater in period t, where $c_{i,t},\ d_{i,t}$ are the periodic groundwater pumpage cost function parameters. The second cost component is the periodic surface water holding cost at user i's reservoir, which is computed as the average holding cost of surface water between periods t and $t+1$, and is given by $C^s_{i,t}(u^s_{i,t}, x^s_{i,t}) = [h_{i,t}(x^s_{i,t+1} + x^s_{i,t})]/2 = 0.5h_{i,t}(2x^s_{i,t} - u^s_{i,t})$, where $x^s_{i,t+1} = x^s_{i,t} - u^s_{i,t}$ and $h_{i,t}$ is the periodic cost of holding one unit of surface water at user i's reservoir.

Eventually, $C_{i,t}(\vec{u}_{i,t}, \vec{x}_{i,t})$ is given by

$$C_{i,t}(\vec{u}_{i,t}, \vec{x}_{i,t}) = c_{i,t}(x^g_{i,0} - x^g_{i,t})u^g_{i,t} + d_{i,t}(u^g_{i,t})^2 + 0.5h_{i,t}(2x^s_{i,t} - u^s_{i,t}) \tag{5.3}$$

From Eqns (5.1)-(5.3), the periodic profit function at user i becomes

$$f_{i,t}(\vec{u}_{i,t}, \vec{x}_{i,t}) = [\rho_{i,t}a_{i,t}(u^g_{i,t} + u^s_{i,t}) - 0.5\rho_{i,t}b_{i,t}(u^g_{i,t} + u^s_{i,t})^2] - [c_{i,t}(x^g_{i,0} - x^g_{i,t})u^g_{i,t} + d_{i,t}(u^g_{i,t})^2 + 0.5h_{i,t}(2x^s_{i,t} - u^s_{i,t})] \tag{5.4}$$

where the cost-revenue parameters are assumed to be positive, non-identical and time-variant. We assume that $f_{i,t}(\vec{u}_{i,t}, \vec{x}_{i,t})$ is jointly concave and positive in $\vec{u}_{i,t},\ i = 1, 2,\ t = 1, 2$. To ensure that, we have the following result

Proposition 5.1 (Strict Concavity) *The function $f_{i,t}(\vec{u}_{i,t}, \vec{x}_{i,t})$ is strictly jointly concave in $\vec{u}_{i,t},\ i = 1, 2,\ t = 1, 2$.*

Proof *See Appendix.*

Next, we examine the decentralized and centralized management problems separately.

5.2 The Decentralized Problem

In the decentralized problem, each user has the objective of maximizing her own total discounted profit by choosing the water usage quantity in each period over a two periods planning horizon. Due to the commonality of the underground aquifer, each user also has to take into account the water usage of her neighbor. This generates a two-player normal-form game, where the water usage quantities in each period are the strategies of a player (a user), and the payoff function is given by a user's total discounted profit over the entire horizon. The strategy space of any user is constructed from the other user's decision of water usage and the available (and finite) water stocks in any period. In this section, we consider this game-theoretic model and investigate its properties.

The decentralized problem ($P1$) can be stated as a dynamic program (DP) as follows. Let $\Gamma^*_{i,t}(\vec{u}_t, \vec{x}_t)$ denote the maximum total profit under an optimal water usage scheme for user i for periods t through the end of horizon, where $\vec{u}_t = (u^g_{1,t}, u^s_{1,t}, u^g_{2,t}, u^s_{2,t})$ denotes the water usage vector for both users in period t and $\vec{x}_t = (x^g_{1,t}, x^s_{1,t}, x^g_{2,t}, x^s_{2,t})$ denotes the water stock vector for both users at the beginning of period t, which gives the state of the system across both users. For $t = 1, 2$ and $i, j = 1, 2,\ i \neq j$, the following DP solves the decentralized problem of user i,

$$(P1) : \Gamma^*_{i,t}(\vec{u}_t, \vec{x}_t) = \max_{\{u^g_{i,t},\ u^s_{i,t}\}} \Gamma_{i,t}(\vec{u}_t, \vec{x}_t) = \max_{\{u^g_{i,t},\ u^s_{i,t}\}} [f_{i,t}(\vec{u}_{i,t}, \vec{x}_{i,t}) + \beta_{i,t}\Gamma^*_{i,t+1}(\vec{u}_{t+1}, \vec{x}_{t+1})] \tag{5.5}$$

$$s.t. \quad x^g_{i,t+1} = x^g_{i,t} + Q_{j,i} - u^g_{i,t} = x^g_{i,t} - (1-\alpha)u^g_{i,t} - \alpha u^g_{j,t} \tag{5.6}$$

$$x^s_{i,t+1} = x^s_{i,t} - u^s_{i,t} \tag{5.7}$$

$$0 \leq u^g_{i,t} \leq x^g_{i,t} \tag{5.8}$$

$$0 \leq u^s_{i,t} \leq x^s_{i,t} \tag{5.9}$$

In the above, the decision variables for the two simultaneous optimization problems are the water usage quantities of each user in each period, $u^g_{i,t}$ and $u^s_{i,t}$. Eqn (5.6) corresponds to the recursive temporal relationship among the groundwater stocks of users as dictated by Darcy's Law. In our formulation, we assume the same finite hydrological transmissivity coefficient α for both users and both periods. Eqn (5.7) gives the recursive temporal relationship governing the surface water stock balance at user i. Eqn (5.8) gives the bounds on each user's groundwater usage quantities whereas Eqn (5.9) gives those on surface water usage quantities. We shall assume the discount rate $\beta_{i,t} = \beta_i$ with $0 \leq \beta_i \leq 1$, for all i and t, and we set $x^g_{i,1} = x^g_1$, $x^s_{i,1} = x^s_1$ and $\Gamma^*_{i,3}(\vec{u}_3, \vec{x}_3) \equiv 0$, for all $\vec{u}_3$ $\vec{x}_3$ and for all i. Furthermore, we take all cost and revenue parameters to be non-identical but time-variant for both users. Similar to the models of Saak and Peterson [52] and Saleh et al. [54], in order to investigate the greedy behavior of users in water consumption, we assume full depletion of ground and surface water by both users over the entire time horizon. The following result presents the condition under which full depletion of water is guaranteed.

Proposition 5.2 (Full Water Depletion) *If* $\rho_{i,t}a_{i,t} > max\{[\rho_{i,t}b_{i,t} + 2d_{i,t} - c_{i,t}]^+ x^g_{i,0} + \rho_{i,t}b_{i,t}x^s_{i,0} + c_{i,t}x^g_{i,0}, \rho_{i,t}b_{i,t}(x^g_{i,0} + x^s_{i,0}) - 0.5h_{i,t}\}$*, the function* $f_{i,t}(\vec{u}_{i,t}, \vec{x}_{i,t})$ *attains its maximum at* $\vec{u}^*_{i,t} = \vec{x}_{i,t}$*,* $i = 1, 2,\ t = 1, 2.$

Proof *See Appendix.*

This result has two implications: *(i)* The myopic solution of the problem is trivial; that is, all water resources are depleted in the first period for any length of the horizon. *(ii)* In the optimal solution, user i depletes her water resources in the very last period, $\vec{u}^*_{i,2} = \vec{x}_{i,2} = (x^g_{i,2}, x^s_{i,2})$, where $x^g_{i,2}$ and $x^s_{i,2}$ are obtained from Eqn (5.6) and Eqn (5.7), respectively, $i = 1, 2$. Therefore, the objective function in Eqn (5.5) becomes $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1) = f_{i,1}(\vec{u}_{i,1}, \vec{x}_{i,1}) + \beta_i f^*_{i,2}(\vec{x}_{i,2}, \vec{x}_{i,2})$, where $f^*_{i,2}(\vec{x}_{i,2}, \vec{x}_{i,2})$ is the optimal profit-to-go from period 2 to period 1. Furthermore, $\vec{x}_{i,2}$ is only a function of $\vec{u}_1$; and, hence, the problem given in (5.5)-(5.9) reduces to a single period problem which is only a function of $\vec{u}_1$ and the initial water stock vector in period 1; $\vec{x}_1$. Below we use these implications to establish certain structural properties and to obtain a tighter formulation of the original problem. First, we establish some properties of $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$ with respect to (w.r.t.) $\vec{u}_1$.

Proposition 5.3 (Concavity) *The function* $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$ *is jointly concave in* $\vec{u}_1$ *if and only if* $\rho_{i,2}b_{i,2} \geq 2[\frac{(1-\alpha)^2+\alpha^2}{(1-\alpha)^3}](c_{i,2} - d_{i,2})$*,* $i = 1, 2.$

Proof *See Appendix.*

The above result guarantees a well-behaved objective function for optimization. In particular, joint concavity of $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$ in $\vec{u}_1$ guarantees that the local and global optima of $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$ are the same, which is of crucial importance to the existence of Nash equilibrium as will be shown later. Thus, the problem $(P1)$ reduces to the following problem $(P2)$. For $i = 1, 2$, we have

$$(P2) : \Gamma^*_{i,1}(\vec{u}_1, \vec{x}_1) = \max_{\{u^g_{i,1},\ u^s_{i,1}\}} \Gamma_{i,1}(\vec{u}_1, \vec{x}_1) = \max_{\{u^g_{i,1},\ u^s_{i,1}\}} [f_{i,1}(\vec{u}_{i,1}, \vec{x}_{i,1}) + \beta_i f^*_{i,2}(\vec{x}_{i,2}, \vec{x}_{i,2})] \tag{5.10}$$

$$s.t. \quad 0 \leq u^g_{i,1} \leq x^g_1 \tag{5.11}$$

$$0 \leq u^s_{i,1} \leq x^s_1 \tag{5.12}$$

where the water stocks in the second period $x^g_{i,2}$ and $x^s_{i,2}$ are given by Eqn (5.6) and Eqn (5.7), respectively. The problem $(P2)$ stated in Eqns (5.10)-(5.12) corresponds to a single period strategic form-game given by the payoff function $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$ and the strategy sets $u^g_{i,1} \in [0, x^g_1]$ and $u^s_{i,1} \in [0, x^s_1]$. We observe that the strategy sets are nonempty, continuous, convex and compact (closed and bounded) and the payoff function is continuous and jointly concave in the players' strategies as given in Proposition 3.3. Hence, from Theorem 1 in Dasgubta and Maskin [16], we have the following result.

Proposition 5.4 (Existence of Nash Equilibrium) *The two-player game which corresponds to the decentralized problem* $(P2)$ *has (at least one) Nash equilibrium.*

In the following discussion, we characterize the optimal policies of water allocation and the Nash equilibria of water usage corresponding to the above decentralized problem. Since we assume that both users start with equal initial stocks of water from both sources and by Proposition 5.2, it is optimal to have, $u^{g*}_{i,2} = x^g_{i,2} = x^g_1 - (1-\alpha)u^g_{i,1} - \alpha u^g_{j,1}$, $i, j = 1, 2,\ i \neq j$, and $u^{s*}_{i,2} = x^s_{i,2} = x^s_1 - u^s_{i,1}$, $i = 1, 2$.

The total discounted cost incurred and the total discounted revenue realized, from water usage over the two-period planning horizon can be written, respectively, as

$C_i(\vec{u}_1, \vec{x}_1) = C_{i,1}(\vec{u}_{i,1}, \vec{x}_{i,1}) + \beta_i C^*_{i,2}(\vec{x}_{i,2}, \vec{x}_{i,2})$ and $R_i(\vec{u}_1, \vec{x}_1) = R_{i,1}(\vec{u}_{i,1}, \vec{x}_{i,1}) + \beta_i R^*_{i,2}(\vec{x}_{i,2}, \vec{x}_{i,2})$.

In this form, $R^*_{i,2}(\vec{x}_{i,2}, \vec{x}_{i,2})$ and $C^*_{i,2}(\vec{x}_{i,2}, \vec{x}_{i,2})$, respectively, represent the optimal revenue-to-go and the optimal cost-to-go functions from period 2 to period 1, for $i = 1, 2$. Accordingly, the objective function in problem $(P2)$ is rewritten as $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1) = R_i(\vec{u}_1, \vec{x}_1) - C_i(\vec{u}_1, \vec{x}_1)$, $i = 1, 2$.

Now, let the total water allocated to user i in period 1 be denoted by $w_{i,1} = u^g_{i,1} + u^s_{i,1} \leq x^g_1 + x^s_1$, $i = 1, 2$. Obviously, we have $u^s_{i,1} = w_{i,1} - u^g_{i,1}$ and $\partial u^g_{i,1}/\partial u^s_{i,1} = \partial u^s_{i,1}/\partial u^g_{i,1} = -1$, $i = 1, 2$. In order to determine the optimal water allocation and usage policy in period 1, without loss of generality, we conduct the analysis on the behavior of $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$ w.r.t. $u^g_{i,1}$. To this end, we substitute $u^s_{i,1} = w_{i,1} - u^g_{i,1}$ in $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$ which becomes as follows

$$\begin{aligned}\Gamma_{i,1}(\vec{u}_1, \vec{x}_1) = {} & [\rho_{i,1} a_{i,1} w_{i,1} - 0.5\rho_{i,1} b_{i,1} w^2_{i,1} + \beta_i \rho_{i,2} a_{i,2}(x^g_1 + x^s_1 - w_{i,1} + \alpha(u^g_{i,1} - u^g_{j,1})) - \\ & 0.5\beta_i \rho_{i,2} b_{i,2}(x^g_1 + x^s_1 - w_{i,1} + \alpha(u^g_{i,1} - u^g_{j,1}))^2] - [d_{i,1}(u^g_{i,1})^2 + 0.5h_{i,1}(2x^s_1 - w_{i,1} + \\ & u^g_{i,1}) + \beta_i c_{i,2}((1-\alpha)u^g_{i,1} + \alpha u^g_{j,1})(x^g_1 - (1-\alpha)u^g_{i,1} - \alpha u^g_{j,1}) + \beta_i d_{i,2}(x^g_1 - (1 - \\ & \alpha)u^g_{i,1} - \alpha u^g_{j,1})^2 + 0.5\beta_i h_{i,2}(x^s_1 - w_{i,1} + u^g_{i,1})].\end{aligned}$$

We observe that the first and second derivatives of $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$ w.r.t. $u^g_{i,1}$ are given, respectively, by

$\partial\Gamma_{i,1}(.,.)/\partial u^g_{i,1} = -[2d_{i,1} + \beta_i \alpha^2 \rho_{i,2} b_{i,2} + 2\beta_i(1-\alpha)^2(d_{i,2} - c_{i,2})]u^g_{i,1} + \beta_i \alpha[\rho_{i,2} a_{i,2} - \rho_{i,2} b_{i,2}(x^g_1 + x^s_1 - w_{i,1} - \alpha u^g_{j,1})] - 0.5(h_{i,1} + \beta_i h_{i,2}) + \beta_i \alpha(1-\alpha)c_{i,2} u^g_{j,1} - \beta_i(1-\alpha)(c_{i,2} - 2d_{i,2})(x^g_1 - \alpha u^g_{j,1})$ and

$\partial^2\Gamma_{i,1}(.,.)/\partial(u^g_{i,1})^2 = -[2d_{i,1} + \beta_i \alpha^2 \rho_{i,2} b_{i,2} + 2\beta_i(1-\alpha)^2(d_{i,2} - c_{i,2})]$, $i = 1, 2$.

The first two derivatives aid us in determining the optimal water allocation polices of user i in period 1 through investigating the behavioral properties of the $\Gamma_{i,1}(.,.)$ w.r.t. groundwater usage in period 1, $u^g_{i,1}$. In particular, we need the first derivative of $\Gamma_{i,1}(.,.)$ w.r.t. $u^g_{i,1}$ to determine the direction (increasing/decreasing) of $\Gamma_{i,1}(.,.)$ w.r.t. $u^g_{i,1}$ at $u^g_{i,1} = 0$. Also, the second derivative of $\Gamma_{i,1}(.,.)$ w.r.t. $u^g_{i,1}$ is employed to determine the shape (convex/concave) of $\Gamma_{i,1}(.,.)$ w.r.t. $u^g_{i,1}$. First, we establish the optimal water allocation polices in period 1 for user i assuming that she knows two pieces of information; her optimal total water usage in period 1 and the water usage (response) of her neighbor (user j) in period 1, as shown in the following result.

Proposition 5.5 (Optimal Water Allocation Policy in Period 1)

(a) For a given total optimal water usage of user i; $w^*_{i,1} = u^{g*}_{i,1} + u^{s*}_{i,1} \leq (x^g_1 + x^s_1)$ *and a given response of user j;* $(u^g_{j,1}, u^s_{j,1})$, $i, j = 1, 2,\ i \neq j$, *the optimal usage policy in period 1 is given by*

(i) If $k_1 < 0$, *then*

$$(u^{g*}_{i,1}, u^{s*}_{i,1}) = \begin{cases} (max\{w^*_{i,1} - x^s_1, \hat{u}^g_{i,1}\}, w^*_{i,1} - u^{g*}_{i,1}) \ or & \\ (min\{w^*_{i,1}, x^g_1\}, w^*_{i,1} - u^{g*}_{i,1}), & if \quad k_2 > 0 \\ (w^*_{i,1} - x^s_1, x^s_1), & if \quad k_2 \leq 0 \end{cases}$$

(ii) If $k_1 > 0$, *then*

$$(u^{g*}_{i,1}, u^{s*}_{i,1}) = \begin{cases} (min\{x^g_1, w^*_{i,1}\}, w^*_{i,1} - u^{g*}_{i,1}), & if \quad k_2 > 0 \\ (max\{0, w^*_{i,1} - x^s_1\}, w^*_{i,1} - u^{g*}_{i,1}) \ or & \\ (min\{x^g_1, w^*_{i,1}\}, w^*_{i,1} - u^{g*}_{i,1}), & if \quad k_2 \leq 0 \end{cases}$$

where $k_1 = \partial^2 \Gamma_{i,1}(.,.)/\partial(u^g_{i,1})^2 = -[2d_{i,1} + \beta_i \alpha^2 \rho_{i,2} b_{i,2} + 2\beta_i(1-\alpha)^2(d_{i,2} - c_{i,2})]$,

$k_2 = (\partial \Gamma_{i,1}(.,.)/\partial u^g_{i,1})|_{(u^g_{i,1}=0)} = \beta_i \alpha[\rho_{i,2} a_{i,2} - \rho_{i,2} b_{i,2}(x^g_1 + x^s_1 - w^*_{i,1} - \alpha u^g_{j,1})] - 0.5(h_{i,1} + \beta_i h_{i,2}) + \beta_i \alpha(1-\alpha) c_{i,2} u^g_{j,1} - \beta_i(1-\alpha)(c_{i,2} - 2d_{i,2})(x^g_1 - \alpha u^g_{j,1})$ *and* $\hat{u}^g_{i,1} = -(k_2/k_1)$.

(b) For a given response of user j; $(u^g_{j,1}, u^s_{j,1})$, *if the total optimal water usage of user i in period 1,* $w^*_{i,1}$, *is such that* $w^*_{i,1} > (x^g_1 + x^s_1)$, $i, j = 1, 2,\ i \neq j$, *then the optimal usage policy in period 1 is given by* $(u^{g*}_{i,1}, u^{s*}_{i,1}) \in \{(0,0), (x^g_1, 0), (0, x^s_1), (x^g_1, x^s_1)\}$.

Proof *See Appendix.*

Having the optimal usage policy determined, we are now ready to find the optimal water usage in period 1; $w^*_{i,1}$. To this end, we optimize $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$ w.r.t. $w^*_{i,1}$ for each candidate solution derived from the optimal usage polices stated in Proposition 5.5, where $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1) = R_i(\vec{u}_1, \vec{x}_1) - C_i(\vec{u}_1, \vec{x}_1)$. The following result presents the optimal water usage solutions in period 1; $w^*_{i,1}$, for all candidate optimal solutions stated in Proposition 5.5.

Proposition 5.6 (Optimal Total Water Usage in Period 1) *(a) For a given response of user j; $(u^g_{j,1}, u^s_{j,1})$, the total optimal water usage of user i in period 1; $w^*_{i,1}$, where $w^*_{i,1} \leq (x^g_1 + x^s_1)$, $i, j = 1, 2,\ i \neq j$, is given as follows*

(i) For $(u^{g*}_{i,1}, u^{s*}_{i,1}) = (x^g_1, w^*_{i,1} - x^g_1)$,

$$w^*_{i,1} = \frac{\rho_{i,1}a_{i,1} - \beta_i\rho_{i,2}a_{i,2} + \beta_i\rho_{i,2}b_{i,2}((1+\alpha)x^g_1 + x^s_1 - \alpha u^g_{j,1}) + 0.5(h_{i,1} + \beta_i h_{i,2})}{\rho_{i,1}b_{i,1} + \beta_i\rho_{i,2}b_{i,2}}.$$

(ii) For $(u^{g*}_{i,1}, u^{s*}_{i,1}) = (0, w^*_{i,1})$,

$$w^*_{i,1} = \frac{\rho_{i,1}a_{i,1} - \beta_i\rho_{i,2}a_{i,2} + \beta_i\rho_{i,2}b_{i,2}(x^g_1 + x^s_1 - \alpha u^g_{j,1}) + 0.5(h_{i,1} + \beta_i h_{i,2})}{\rho_{i,1}b_{i,1} + \beta_i\rho_{i,2}b_{i,2}}.$$

(iii) For $(u^{g*}_{i,1}, u^{s*}_{i,1}) = (\hat{u}^g_{i,1}, w^*_{i,1} - \hat{u}^g_{i,1})$, $w^*_{i,1} = \tilde{k_2}/\tilde{k_1}$ and $\hat{u}^g_{i,1} = \gamma_0 + \gamma_1 w^*_{i,1}$, where

$\tilde{k_1} = k_1\rho_{i,1}b_{i,1} - \beta_i(\beta_i\alpha^2\rho_{i,2}b_{i,2} + k_1)\rho_{i,2}b_{i,2}(\alpha\gamma_1 - 1) - 2\beta_i\alpha\gamma_1\rho_{i,2}b_{i,2}d_{i,1} + 2(\beta_i)^2\alpha(1-\alpha)^2\gamma_1\rho_{i,2}b_{i,2}d_{i,1}(c_{i,2} - d_{i,2})$,

$\tilde{k_2} = k_1\rho_{i,1}a_{i,1} + \beta_i\rho_{i,2}b_{i,2}(\beta_i\alpha^2\rho_{i,2}b_{i,2} + k_1)[x^g_1 + x^s_1 - \alpha u^g_{j,1} - \alpha\gamma_0 - 1] + 2\beta_i\alpha\gamma_0\rho_{i,2}b_{i,2}d_{i,1} + 0.5(h_{i,1} + \beta_i h_{i,2})(\beta_i\alpha\rho_{i,2}b_{i,2} + k_1) + \beta_i^2\alpha(1-\alpha)\rho_{i,2}b_{i,2}(c_{i,2} - 2d_{i,2})(x^g_1 - \alpha u^g_{j,1}) - 2\beta_i^2\alpha(1-\alpha)^2\rho_{i,2}b_{i,2}c_{i,2}(c_{i,2} - d_{i,2})\gamma_0 - \beta_i^2\alpha^2(1-\alpha)\rho_{i,2}b_{i,2}c_{i,2}u^g_{j,1}$,

$\gamma_0 = [\beta_i\alpha(\rho_{i,2}a_{i,2} - \beta_i\rho_{i,2}b_{i,2}(x^g_1 + x^s_1 - \alpha u^g_{j,1})) - 0.5(h_{i,1} + \beta_i h_{i,2}) + \beta_i\alpha(1-\alpha)c_{i,2}u^g_{j,1} - \beta_i(1-\alpha)(c_{i,2} - 2d_{i,2})(x^g_1 - \alpha u^g_{j,1})]/[-k_1]$, $\gamma_1 = [\beta_i\alpha\rho_{i,2}b_{i,2}]/[-k_1]$, *$k_1$ and k_2 are as defined before in Proposition 3.5.*

(iv) For $(u^{g*}_{i,1}, u^{s*}_{i,1}) = (w^*_{i,1}, 0)$, $w^*_{i,1} = \xi_1/\xi_0$, where

$\xi_1 = \rho_{i,1}a_{i,1} - \beta_i(1-\alpha)\rho_{i,2}a_{i,2} + \beta_i(1-\alpha)\rho_{i,2}b_{i,2}(x^g_1 + x^s_1 - \alpha u^g_{j,1}) + \beta_i\alpha(1-\alpha)c_{i,2}u^g_{j,1} - \beta_i(1-\alpha)(c_{i,2} - 2d_{i,2})(x^g_1 - \alpha u^g_{j,1})$ *and*

$\xi_0 = \rho_{i,1}b_{i,1} + \beta_i(1-\alpha)^2\rho_{i,2}b_{i,2} + 2d_{i,1} - 2\beta_i(1-\alpha)^2(c_{i,2} - d_{i,2})$

(v) *For* $(u^{g*}_{i,1}, u^{s*}_{i,1}) = (w^*_{i,1} - x^s_1, x^s_1)$, $w^*_{i,1} = \xi_2/\xi_0$, *where*

$\xi_2 = \rho_{i,1}a_{i,1} - \beta_i(1-\alpha)\rho_{i,2}a_{i,2} + \beta_i(1-\alpha)[\rho_{i,2}b_{i,2} - (c_{i,2} - 2d_{i,2})](x^g_1 + (1-\alpha)x^s_1 - \alpha u^g_{j,1}) + 2d_{i,1}x^s_1 + \beta_i\alpha(1-\alpha)c_{i,2}u^g_{j,1}$

(b) *If the total optimal water usage of user* i *in period 1,* $w^*_{i,1}$*, is infeasible (i.e.* $w^*_{i,1} > (x^g_1 + x^s_1)$*), then the optimal solution in period 1 is given by* $(u^{g*}_{i,1}, u^{s*}_{i,1}) \in \{(0,0), (x^g_1, 0), (0, x^s_1), (x^g_1, x^s_1)\}$, $i = 1, 2$.

Proof *See Appendix.*

Identical Users: where both users have the same, but time-variant, revenue-cost parameters (*i.e.* $\rho_{i,t} = \rho_t,\ a_{i,t} = a_t,\ b_{i,t} = b_t,\ c_{i,t} = c_t,\ d_{i,t} = d_t$ and $h_{i,t} = h_t$ for all i and t) and the same discount rate (i.e. $\beta_i = \beta$, for all i. Under the identical setting, in accordance with Propositions 5.4 and 5.5, we obtain the optimal Nash equilibria of water usage in period 1 which is turned to be symmetric across users as shown in the following result.

Corollary 5.1 (Optimal Nash Equilibria of Water Usage in Period 1)

(a) *The optimal Nash equilibria of water usage in period 1;* w^*_1*, where* $w^*_1 \le (x^g_1 + x^s_1)$*, are symmetric across users and given as follows*

(i) *For* $(u^{g*}_1, u^{s*}_1) \in \{(x^g_1, w^*_1 - x^g_1), (0, w^*_1)\}$,

$$w^*_1 = \frac{\rho_1 a_1 - \beta\rho_2 a_2 + \beta\rho_2 b_2(x^g_1 + x^s_1) + 0.5(h_1 + \beta h_2)}{\rho_1 b_1 + \beta\rho_2 b_2}.$$

(ii) *For* $(u^{g*}_1, u^{s*}_1) = (\hat{u}^g_1, w^*_1 - \hat{u}^g_1)$,

$$w^*_1 = \frac{\rho_1 a_1 - \beta\rho_2 a_2 + \beta\rho_2 b_2(x^g_1 + x^s_1) - 0.5(h_1 + \beta h_2)(\epsilon_1 - 1) + \epsilon_0\epsilon_1(\beta\alpha c_2 - 2d_1) - \beta\epsilon_1(c_2 - 2d_2)(x^g_1 - \epsilon_0)}{\rho_1 b_1 + \beta\rho_2 b_2 + 2\epsilon_1^2(d_1 - \beta\alpha c_2 - \beta d_2)}$$

and $\hat{u}^g_1 = \epsilon_0 + \epsilon_1 w^*_1$*, where*

$$\epsilon_0 = \frac{\beta\alpha[\rho_2 a_2 - \rho_2 b_2(x^g_1 + x^s_1)] - 0.5(h_1 + \beta h_2) - \beta(1-\alpha)(c_2 - 2d_2)x^g_1}{2d_1 + 2\beta(1-\alpha)(d_2 - c_2)} \text{ and } \epsilon_1 = \frac{\beta\rho_2 b_2}{2d_1 + 2\beta(1-\alpha)(d_2 - c_2)}.$$

(iii) *For* $(u^{g*}_1, u^{s*}_1) = (w^*_1, 0)$,

$$w^*_1 = \frac{\rho_1 a_1 - \beta(1-\alpha)\rho_2 a_2 + \beta(1-\alpha)\rho_2 b_2(x^g_1 + x^s_1) - \beta(1-\alpha)(c_2 - 2d_2)x^g_1}{\rho_1 b_1 + \beta(1-\alpha)\rho_2 b_2 + 2d_1 - 2\beta(1-\alpha)(c_2 - d_2)}.$$

(*iv*) *For* $(u_1^{g*}, u_1^{s*}) = (w_1^* - x_1^s, x_1^s)$,

$$w_1^* = \frac{\rho_1 a_1 - \beta(1-\alpha)\rho_2 a_2 + \beta(1-\alpha)[\rho_2 b_2 - (c_2 - 2d_2)](x_1^g + x_1^s) + [2d_1 - \beta\alpha(1-\alpha)c_2]x_1^s}{\rho_1 b_1 + \beta(1-\alpha)\rho_2 b_2 + 2d_1 - 2\beta(1-\alpha)(c_2 - d_2)}.$$

(*b*) *If the total optimal water usage of in period 1,* w_1^**, is infeasible (i.e.* $w_1^* > (x_1^g + x_1^s)$*), then the optimal Nash equilibria of water usage in period 1 is given by* $(u_1^{g*}, u_1^{s*}) \in \{(0,0), (x_1^g, 0), (0, x_1^s), (x_1^g, x_1^s)\}$.

The strategic interaction between users in the strategic form-game corresponding to the decentralized problem is analyzed and represented by the Nash equilibria given above. The Nash equilibria correspond to the simultaneous solution of the two constrained decentralized optimization problems given in (5.10)-(5.12). At a Nash equilibrium no user deviates unilaterally from her water usage strategy in period 1 as non of users would gain from such a deviation. In other words, if a strategy of water usage in period 1 is selected by user i and no user would gain by a unilateral deviation from that strategy while user j's water usage strategy in period 1 does not change, then the current set of water usage strategies in period 1 and the corresponding total usage profits from a Nash equilibrium. Since Corollary 5.1 presents all possible optimal solutions of $(P2)$, then the Nash equilibria are among those solutions stated above in $(i) - (v)$ and (b), where the optimal Nash equilibrium is the one that gives the greatest value of $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$. Recall that in Proposition 5.4, we state that the decentralized problem has at least one Nash equilibrium and, hence, users might posses multiple optimal Nash equilibrium. The optimal Nash equilibria in part (a) of the above result depend on the aquifer transmissivity coefficient, α. Therefore, the hydrological properties of the groundwater aquifer's, summarized by α, play an essential role in the actions of users playing the strategic form-game corresponding to the decentralized problem, even when users are identical. However, under the identical users setting, the transmissivity coefficient α does not have any impact on the users actions in the centralized problem, as will be shown later in the following section.

5.3 The Centralized Problem

In the centralized setting, we envision a central decision maker (social planner in the public policy parlance) aiming at determining the optimal water usage for each user so that the total joint discounted profit of both users over the course of two periods is maximized. Let $\tilde{\Gamma}^*_t(\vec{u}_t, \vec{x}_t)$ denote the maximum total joint profit under optimal water usage schemes for both users for period t until the end of horizon. The centralized problem $(P3)$ can be stated as a DP as follows. For $t = 1, 2$ and $i = 1, 2$, we have

$$(P3) : \tilde{\Gamma}^*_t(\vec{u}_t, \vec{x}_t) = \max_{\{u^g_{i,t}, u^s_{i,t}\}} \tilde{\Gamma}_t(\vec{u}_t, \vec{x}_t) = \max_{\{u^g_{i,t}, u^s_{i,t}\}} \{\sum_{i=1}^{2} f_{i,t}(\vec{u}_{i,t}, \vec{x}_{i,t}) + \beta_t \tilde{\Gamma}^*_{t+1}(\vec{u}_{t+1}, \vec{x}_{t+1})\} \tag{5.13}$$

$$s.t. \quad (5.6) - (5.9)$$

We assume that the social planner's discount rate $\beta_t = \beta$ with $0 \leq \beta \leq 1$. We retain all other conventions and notations of the decentralized problem. Proposition 5.2 implies that in period 2, we have $\vec{u}^*_{i,2} = \vec{x}_{i,2} = (x^g_{i,2}, x^s_{i,2})$, $i = 1, 2$, in this problem as well. Therefore, in $(P3)$, we have $\tilde{\Gamma}_t(\vec{u}_t, \vec{x}_t) = \tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1) = \sum_{i=1}^{2}\{f_{i,1}(\vec{u}_{i,1}, \vec{x}_{i,1}) + \beta f^*_{i,2}(\vec{x}_{i,2}, \vec{x}_{i,2})\}$. In accordance with Proposition 5.3, we have the following result.

Corollary 5.2 (Concavity) *The function $\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1)$ is jointly concave in $\vec{u}_1$ if and only if $\rho_{i,2} b_{i,2} \geq 2[\frac{(1-\alpha)^2+\alpha^2}{(1-\alpha)^3}](c_{i,2} - d_{i,2})$, $i = 1, 2$.*

Thus, the centralized problem $(P3)$ reduces to the following problem $(P4)$. For $t = 1, 2$ and $i, j = 1, 2,\ i \neq j$, we have

$$(P4) : \tilde{\Gamma}^*_1(\vec{u}_1, \vec{x}_1) = \max_{\{u^g_{i,1}, u^s_{i,1}\}} \tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1) = \max_{\{u^g_{i,1}, u^s_{i,1}\}} \{\sum_{i=1}^{2} f_{i,1}(\vec{u}_{i,1}, \vec{x}_{i,1}) + \beta f^*_{i,2}(\vec{x}_{i,2}, \vec{x}_{i,2})\} \tag{5.14}$$

$$s.t. \ \ (5.11) - (5.12)$$

Analogous to the analysis in the decentralized problem, we consider the total discounted cost incurred over the two-period planning horizon given by $\tilde{C}_1(\vec{u}_1, \vec{x}_1) = \sum_{i=1}^{2}\{C_{i,1}(\vec{u}_{i,1}, \vec{x}_{i,1}) + \beta C^*_{i,2}(\vec{x}_{i,2}, \vec{x}_{i,2})\}$. Similarly, the total discounted revenue over the two-period planning horizon is given by

$\tilde{R}_1(\vec{u}_1, \vec{x}_1) = \sum_{i=1}^{2}\{R_{i,1}(\vec{u}_{i,1}, \vec{x}_{i,1}) + \beta R^*_{i,2}(\vec{x}_{i,2}, \vec{x}_{i,2})\}$. To find the optimal water usage in period 1; $w^*_{i,1}$, we optimize $\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1)$ w.r.t. $w_{i,1}$, where $\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1) = \tilde{R}_1(\vec{u}_1, \vec{x}_1) - \tilde{C}_1(\vec{u}_1, \vec{x}_1)$. Again, similar to the analysis of the decentralized problem, our analysis here is also conducted in accordance with the behavior of $\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1)$ w.r.t. $u^g_{i,1}$ as well. More formally, let $w_{i,1} = u^g_{i,1} + u^s_{i,1} \leq (x^g_1 + x^s_1)$ be the total water usage of user i in period 1, then we have $\partial u^g_{i,1}/\partial u^s_{i,1} = \partial u^s_{i,1}/\partial u^g_{i,1} = -1$, for $i = 1, 2$. After substituting $u^g_{i,1} = w_{i,1} - u^s_{i,1}$ in $\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1)$, we get, for $i, j = 1, 2, \ i \neq j$,

$$\begin{aligned}\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1) = {} & \rho_{i,1}a_{i,1}w_{i,1} - 0.5\rho_{i,1}b_{i,1}w^2_{i,1} + \rho_{j,1}a_{j,1}w_{j,1} - 0.5\rho_{j,1}b_{j,1}w^2_{j,1} + \beta[\rho_{i,2}a_{i,2}(x^g_1 + \\ & x^s_1 - w_{i,1} + \alpha(u^g_{i,1} - u^g_{j,1})) - 0.5\rho_{i,2}b_{i,2}(x^g_1 + x^s_1 - w_{i,1} + \alpha(u^g_{i,1} - u^g_{j,1}))^2] + \\ & \beta[\rho_{,2}a_{j,2}(x^g_1 + x^s_1 - w_{j,1} + \alpha(u^g_{j,1} - u^g_{i,1})) - 0.5\rho_{j,2}b_{j,2}(x^g_1 + x^s_1 - w_{j,1} + \alpha(u^g_{j,1} - \\ & u^g_{i,1}))^2] - [d_{i,1}(u^g_{i,1})^2 + 0.5h_{i,1}(2x^s_1 - w_{i,1} + u^g_{i,1})] - [d_{j,1}(u^g_{j,1})^2 + 0.5h_{j,1}(2x^s_1 - \\ & w_{j,1} + u^g_{j,1})] - \beta[c_{i,2}((1-\alpha)u^g_{i,1} + \alpha u^g_{j,1})(x^g_1 - (1-\alpha)u^g_{i,1} - \alpha u^g_{j,1}) + d_{i,2}(x^g_1 - \\ & (1-\alpha)u^g_{i,1} - \alpha u^g_{j,1})^2 + 0.5h_{i,2}(x^s_1 - w_{i,1} + u^g_{i,1})] - \beta[c_{j,2}((1-\alpha)u^g_{j,1} + \alpha u^g_{i,1})(x^g_1 - \\ & (1-\alpha)u^g_{j,1} - \alpha u^g_{i,1}) + d_{j,2}(x^g_1 - (1-\alpha)u^g_{j,1} - \alpha u^g_{i,1})^2 + 0.5h_{j,2}(x^s_1 - w_{j,1} + u^g_{j,1})].\end{aligned}$$

We observe that the first and second derivatives of $\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1)$ w.r.t. $u^g_{i,1}$ are given, respectively, by

$\partial\tilde{\Gamma}_1(.,.)/\partial u^g_{i,1} = -[2d_{i,1} + \beta\alpha^2(\rho_{i,2}b_{i,2} + \rho_{j,2}b_{j,2}) + 2\beta(1-\alpha)^2(d_{i,2} - c_{i,2}) + 2\beta\alpha^2(d_{j,2} - c_{j,2})]u^g_{i,1} + \beta\alpha[(\rho_{i,2}a_{i,2} - \rho_{j,2}a_{j,2}) - \rho_{i,2}b_{i,2}(x^g_1 + x^s_1 - w_{i,1} - \alpha u^g_{j,1}) + \rho_{j,2}b_{j,2}(x^g_1 + x^s_1 - w_{j,1} + \alpha u^g_{j,1})] - 0.5(h_{i,1} + \beta_i h_{i,2}) + \beta\alpha(1-\alpha)(c_{i,2} + c_{j,2})u^g_{j,1} - \beta(1-\alpha)(c_{i,2} - 2d_{i,2})(x^g_1 - \alpha u^g_{j,1}) - \beta\alpha(c_{j,2} - 2d_{j,2})(x^g_1 - (1-\alpha)u^g_{j,1})$ and $\partial^2\tilde{\Gamma}_1(.,.)/\partial(u^g_{i,1})^2 = -[2d_{i,1} + \beta\alpha^2(\rho_{i,2}b_{i,2} + \rho_{j,2}b_{j,2}) + 2\beta(1-\alpha)^2(d_{i,2} - c_{i,2}) + 2\beta\alpha^2(d_{j,2} - c_{j,2})]$, $i, j = 1, 2, \ i \neq j$.

The first two derivatives aid us in determining the optimal water allocation polices of user i in period 1 through investigating the behavioral properties of the $\partial\tilde{\Gamma}_1(.,.)/\partial u^g_{i,1}$ w.r.t. groundwater usage in period 1; $u^g_{i,1}$. Namely, we need the first derivative; $\partial\tilde{\Gamma}_1(.,.)/\partial u^g_{i,1}$, to determine the direction (increasing/decreasing) of $\tilde{\Gamma}_1(.,.)$ w.r.t. $u^g_{i,1}$ at $u^g_{i,1} = 0$. Also, the second derivative of $\tilde{\Gamma}_1(.,.)$ w.r.t. $u^g_{i,1}$ is employed to determine the shape (convex/concave) of $\tilde{\Gamma}_1(.,.)$ w.r.t. $u^g_{i,1}$. First, we present the optimal water usage and allocation policy in period 1 for problem $(P4)$ based on the behavior of $\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1)$ w.r.t. $u^g_{i,1}$.

Proposition 5.7 (Optimal Water Allocation Policy in Period 1)

(a) *For a given total optimal water usage of user i;* $w^*_{i,1} = u^{g*}_{i,1} + u^{s*}_{i,1} \leq (x^g_1 + x^s_1)$, $i = 1, 2$, *the optimal usage policy in period 1 is given by*

(i) If $k_1 < 0$, *then*

$$(u^{g*}_{i,1}, u^{s*}_{i,1}) = \begin{cases} (max\{w^*_{i,1} - x^s_1, \hat{u}^g_{i,1}\}, w^*_{i,1} - u^{g*}_{i,1}) \text{ or} & \\ (min\{w^*_{i,1}, x^g_1\}, w^*_{i,1} - u^{g*}_{i,1}), & if \quad k_2 > 0 \\ (w^*_{i,1} - x^s_1, x^s_1), & if \quad k_2 \leq 0 \end{cases}$$

(ii) If $k_1 > 0$, *then*

$$(u^{g*}_{i,1}, u^{s*}_{i,1}) = \begin{cases} (min\{x^g_1, w^*_{i,1}\}, w^*_{i,1} - u^{g*}_{i,1}), & if \quad k_2 > 0 \\ (max\{0, w^*_{i,1} - x^s_1\}, w^*_{i,1} - u^{g*}_{i,1}) \text{ or} & \\ (min\{x^g_1, w^*_{i,1}\}, w^*_{i,1} - u^{g*}_{i,1}), & if \quad k_2 \leq 0 \end{cases}$$

where $k_1 = \partial^2\tilde{\Gamma}_1(.,.)/\partial(u^g_{i,1})^2 = -[2d_{i,1} + \beta\alpha^2(\rho_{i,2}b_{i,2} + \rho_{j,2}b_{j,2}) + 2\beta(1-\alpha)^2(d_{i,2} - c_{i,2}) + 2\beta\alpha^2(d_{j,2} - c_{j,2})]$,

$k_2 = (\partial\tilde{\Gamma}_1(.,.)/\partial u^g_{i,1})|_{(u^g_{i,1}=0)} = \beta\alpha[(\rho_{i,2}a_{i,2} - \rho_{j,2}a_{j,2}) - \rho_{i,2}b_{i,2}(x^g_1 + x^s_1 - w_{i,1} - \alpha u^g_{j,1}) + \rho_{j,2}b_{j,2}(x^g_1 + x^s_1 - w_{j,1} + \alpha u^g_{j,1})] - 0.5(h_{i,1} + \beta_i h_{i,2}) + \beta\alpha(1-\alpha)(c_{i,2} + c_{j,2})u^g_{j,1} - \beta(1-\alpha)(c_{i,2} - 2d_{i,2})(x^g_1 - \alpha u^g_{j,1}) - \beta\alpha(c_{j,2} - 2d_{j,2})(x^g_1 - (1-\alpha)u^g_{j,1})$ *and* $\hat{u}^g_{i,1} = -(k_2/k_1)$, $i, j = 1, 2$, $i \neq j$.

(b) *If the total optimal water usage of user i in period 1,* $w^*_{i,1}$, *is such that* $w^*_{i,1} > (x^g_1 + x^s_1)$, *then the optimal usage policy in period 1 is given by* $(u^{g*}_{i,1}, u^{s*}_{i,1}) \in \{(0,0), (x^g_1, 0), (0, x^s_1), (x^g_1, x^s_1)\}$, $i = 1, 2$.

Proof *See Appendix.*

We observe that both decentralized and centralized problems have the same structure of the optimal water allocation policies of water usage by user i in period 1, but with different parameters settings under each problem. Nonetheless, in the decentralized problem, users experience a strategic form-game as they manage and control their water usage individually. Due to the nature of this game, each user maximizes her own total profit assuming that the water usage (response, considered as a parameter) of her neighbor is known. Eventually, the policies in Proposition 5.5 are stated for a given total water usage of user i and a given response of user j. This is not the case in the centralized problem, where users experience no games since the water usage is managed and controlled centrally by the social planner. In other words, a single problem is needed to be optimized under this setting. Therefore, the policies in Proposition 5.7 are stated for a given total water usage of user i only as the usage of user j is a decision variable and not a given response.

Optimal water usages are found through optimizing $\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1) = \tilde{R}_1(\vec{u}_1, \vec{x}_1) - \tilde{C}_1(\vec{u}_1, \vec{x}_1)$ w.r.t. $w_{i,1}$ for all potential optimal solutions derived from the optimal polices of Proposition 5.7. The following result presents the optimal water usage of user i in period 1 given as a function of water usage (a decision variable) of user j in period 1.

Proposition 5.8 (Optimal Total Water Usage in Period 1)

(a) The total optimal water usage of user i in period 1; $w^*_{i,1}$, where $w^*_{i,1} \leq (x^g_1 + x^s_1)$, for $i, j = 1, 2,\ i \neq j$, is given as follows

(i) For $(u^{g*}_{i,1}, u^{s*}_{i,1}) = (x^g_1, w^*_{i,1} - x^g_1)$,

$$w^*_{i,1} = \frac{\rho_{i,1}a_{i,1} - \beta\rho_{i,2}a_{i,2} + \beta\rho_{i,2}b_{i,2}((1+\alpha)x^g_1 + x^s_1 - \alpha u^g_{j,1}) + 0.5(h_{i,1} + \beta h_{i,2})}{\rho_{i,1}b_{i,1} + \beta\rho_{i,2}b_{i,2}}.$$

(ii) For $(u^{g*}_{i,1}, u^{s*}_{i,1}) = (0, w^*_{i,1})$,

$$w^*_{i,1} = \frac{\rho_{i,1}a_{i,1} - \beta\rho_{i,2}a_{i,2} + \beta\rho_{i,2}b_{i,2}(x^g_1 + x^s_1 - \alpha u^g_{j,1}) + 0.5(h_{i,1} + \beta h_{i,2})}{\rho_{i,1}b_{i,1} + \beta\rho_{i,2}b_{i,2}}.$$

(iii) For $(u^{g*}_{i,1}, u^{s*}_{i,1}) = (\hat{u}^g_{i,1}, w^*_{i,1} - \hat{u}^g_{i,1})$,

$w^*_{i,1} = \tilde{k}_2/\tilde{k}_1$ *and* $\hat{u}^g_{i,1} = \tilde{\gamma}_0 + \tilde{\gamma}_1 w^*_{i,1}$, *where*

$\tilde{k}_1 = \rho_{i,1}b_{i,1} + \beta(\alpha\tilde{\gamma}_1 - 1)^2\rho_{i,2}b_{i,2} + \beta(\alpha\tilde{\gamma}_1)^2\rho_{j,2}b_{j,2} + 2(\tilde{\gamma}_1)^2 d_{i,1} + 2\beta(1 - \alpha\tilde{\gamma}_1)^2(d_{i,2} - c_{i,2}) + 2\beta(\alpha\tilde{\gamma}_1)^2(d_{j,2} - c_{j,2})$,

$\tilde{k}_2 = \rho_{i,1}a_{i,1} + \beta(\alpha\tilde{\gamma}_1 - 1)\rho_{i,2}a_{i,2} - \beta\alpha\tilde{\gamma}_1\rho_{j,2}a_{j,2} - \beta(\alpha\tilde{\gamma}_1 - 1)\rho_{i,2}b_{i,2}(x^g_1 + x^s_1 + \alpha\tilde{\gamma}_0) + \beta\alpha\tilde{\gamma}_1\rho_{j,2}b_{j,2}(x^g_1 + x^s_1 - w_{j,1} - \alpha\tilde{\gamma}_0 + \alpha u^g_{j,1}) - 2\tilde{\gamma}_0\tilde{\gamma}_1 d_{i,1} - 0.5(\tilde{\gamma}_1 - 1)(h_{i,1} + \beta h_{i,2} + \beta(1-\alpha)\tilde{\gamma}_1 c_{i,2}((1-\alpha)\tilde{\gamma}_0 + \alpha u^g_{j,1}) + \beta\alpha\tilde{\gamma}_1 c_{j,2}(\alpha\tilde{\gamma}_0 + (1 - \alpha)u^g_{j,1}) - \beta(1-\alpha)\tilde{\gamma}_1(c_{i,2} - 2d_{i,2})(x^g_1 - (1-\alpha)\tilde{\gamma}_0 - \alpha u^g_{j,1}) - \beta\alpha\tilde{\gamma}_1(c_{j,2} - 2d_{j,2})(x^g_1 - \alpha\tilde{\gamma}_0 - (1-\alpha)u^g_{j,1})$,

$\tilde{\gamma}_0 = [\beta\alpha[(\rho_{i,2}a_{i,2} - \rho_{j,2}a_{j,2}) - \rho_{i,2}b_{i,2}(x^g_1 + x^s_1 - \alpha u^g_{j,1}) + \rho_{j,2}b_{j,2}(x^g_1 + x^s_1 - w_{j,1} + \alpha u^g_{j,1})] - 0.5(h_{i,1} + \beta h_{i,2}) + \beta\alpha(1-\alpha)(c_{i,2} + c_{j,2})u^g_{j,1} - \beta(1 - \alpha)(c_{i,2} - 2d_{i,2})(x^g_1 - \alpha u^g_{j,1}) - \beta\alpha(c_{j,2} - 2d_{j,2})(x^g_1 - (1-\alpha)u^g_{j,1})]/[-k_1]$,

$\tilde{\gamma}_1 = [\beta\alpha\rho_{i,2}b_{i,2}]/[-k_1]$, k_1 *and* k_2 *are as defined before in Proposition 3.7.*

(iv) For $(u^{g*}_{i,1}, u^{s*}_{i,1}) = (w^*_{i,1}, 0)$, $w^*_{i,1} = \theta/\gamma$, *where*

$\theta = \rho_{i,1}a_{i,1} - \beta[(1-\alpha)\rho_{i,2}a_{i,2} + \alpha\rho_{j,2}a_{j,2}] + \beta(1-\alpha)\rho_{i,2}b_{i,2}(x^g_1 + x^s_1 - \alpha u^g_{j,1}) + \beta\alpha\rho_{j,2}b_{j,2}(x^g_1 + x^s_1 - (1-\alpha)u^g_{j,1} - u^s_{j,1}) + \beta\alpha(1-\alpha)(c_{i,2} + c_{j,2})u^g_{j,1} - \beta(1-\alpha)(c_{i,2} - 2d_{i,2})(x^g_1 - \alpha u^g_{j,1}) - \beta\alpha(c_{j,2} - 2d_{j,2})(x^g_1 - (1-\alpha)u^g_{j,1})$ *and*

$\gamma = \rho_{i,1}b_{i,1} + \beta[(1-\alpha)^2\rho_{i,2}b_{i,2} + \alpha^2\rho_{j,2}b_{j,2}] + 2d_{i,1} - 2\beta[(1-\alpha)^2(c_{i,2} - d_{i,2}) + \alpha^2(c_{j,2} - d_{j,2})]$.

(v) For $(u^{g*}_{i,1}, u^{s*}_{i,1}) = (w^*_{i,1} - x^s_1, x^s_1)$, $w^*_{i,1} = \sigma/\gamma$, *where*

$\sigma = \rho_{i,1}a_{i,1} - \beta[(1-\alpha)\rho_{i,2}a_{i,2} + \alpha\rho_{j,2}a_{j,2}] + \beta(1-\alpha)\rho_{i,2}b_{i,2}(x^g_1 + (1 + \alpha)x^s_1 - \alpha u^g_{j,1}) + \beta\alpha\rho_{j,2}b_{j,2}(x^g_1 + x^s_1 + \alpha x^s_1 - (1-\alpha)u^g_{j,1} - u^s_{j,1}) + d_{i,1}x^s_1 + \beta(1-\alpha)c_{i,2}(\alpha u^g_{j,1} - (1-\alpha)x^s_1) + \beta\alpha c_{j,2}((1-\alpha)u^g_{j,1} - \alpha x^s_1) - \beta(1-\alpha)(c_{i,2} - 2d_{i,2})(x^g_1 + (1-\alpha)x^s_1 - \alpha u^g_{j,1}) - \beta\alpha(c_{j,2} - 2d_{j,2})(x^g_1 + \alpha x^s_1 - (1-\alpha)u^g_{j,1})$.

(b) If the total optimal water usage of user i *in period 1,* $w^*_{i,1}$*, is infeasible (i.e.* $w^*_{i,1} > (x^g_1 + x^s_1)$*), then the optimal solution in period 1 is given by* $(u^{g*}_{i,1}, u^{s*}_{i,1}) \in \{(0,0), (x^g_1, 0), (0, x^s_1), (x^g_1, x^s_1)\}$, $i = 1, 2$.

Proof *See Appendix.*

Recall that Proposition 5.6 states the optimal total water usage of user i in period 1, assuming that the water usage (response, considered as a parameter) of user j in period 1 is known by user i. Since each user optimizes her own problem simultaneously with her neighbor, the solution given in Proposition 5.6 is valid for user j as well. However, the solution given in Proposition 5.8 of the centralized problem is stated for user i as a function of user j's usage in period 1, which is still a decision variable that needs to be determined. Therefore, with non-identical users, it is hard to find the optimal total water usage of user i in period 1. More specifically, for each optimal solution of user i, there are 9 candidate optimal solutions (5 solutions stated in $(i)-(v)$ and 4 in (b)) of user j to be substituted in order to get the optimal solution of user i in period 1. Since this is true for any optimal solution, totally, we have to search for the optimal solution among a set of $(9 \times 9 = 81)$ possible combinations! In accordance with this, it is quite easier to look for the optimal solution of the centralized problem when both users are identical. Below, we present the optimal equilibrium solution of water usage in period 1 for the identical users case.

Corollary 5.3 (Optimal Equilibrium Water Usage in Period 1)

(a) The optimal water usage in period 1, w_1^*, where $w_1^* \leq (x_1^g + x_1^s)$, is symmetric across users and is given by

(i) For $(u_1^{g*}, u_1^{s*}) \in \{(x_1^g, w_1^* - x_1^g), (0, w_1^*), (\hat{u}_1^g, w_1^* - \hat{u}_1^g)\}$,

$$w_1^* = \frac{\rho_1 a_1 - \beta\rho_2 a_2 + \beta\rho_2 b_2(x_1^g + x_1^s) + 0.5(h_1 + \beta h_2)}{\rho_1 b_1 + \beta\rho_2 b_2}, \text{ where } \hat{u}_1^g = \frac{-\beta(c_2 - 2d_2)x_{i,1}^g - 0.5(h_1 + \beta h_2)}{2[d_1 - \beta(c_2 - d_2)]}.$$

(ii) For $(u_1^{g*}, u_1^{s*}) = (w_1^*, 0)$, $w_1^* = \frac{\rho_1 a_1 - \beta\rho_2 a_2 + \beta\rho_2 b_2(x_1^g + x_1^s) - \beta(c_2 - 2d_2)x_1^g}{\rho_1 b_1 + \beta\rho_2 b_2 + 2d_1 - 2\beta(c_2 - d_2)}$.

(iii) For $(u_1^{g*}, u_1^{s*}) = (w_1^* - x_1^s, x_1^s)$,

$$w_1^* = \frac{\rho_1 a_1 - \beta\rho_2 a_2 + \beta[\rho_2 b_2 - (c_2 - 2d_2)](x_1^g + x_1^s) + (2d_1 - \beta c_2)x_1^s}{\rho_1 b_1 + \beta\rho_2 b_2 + 2d_1 - 2\beta(c_2 - d_2)}.$$

(b) If the total optimal water usage of in period 1, w_1^*, is infeasible (i.e. $w_1^* > (x_1^g + x_1^s)$), then the optimal solution in period 1 is given by $(u_1^{g*}, u_1^{s*}) \in \{(0,0), (x_1^g, 0), (0, x_1^s), (x_1^g, x_1^s)\}$.

Proof *See Appendix.*

It is obvious that the centralized solution in Corollary 5.3 is unique, symmetric and independent of the transmissivity coefficient α. In other words, when users are identical, regardless of the groundwater aquifer's hydrologic activity along time, users' water usage in both periods is not affected by α when the conjunctive water use is managed centrally by the social planner. However, this is not the case when identical users manage their ground and surface water usage individually in a non-cooperative fashion. More specifically, the two optimal solutions $(w_1^*, 0)$ and $(w_1^* - x_1^s, x_1^s)$ and depend on the value of α, whereas the remaining optimal solutions are independent of α, as shown previously in Corollary 5.1. Accordingly, comparing the optimal Nash equilibria of water usage in Corollary 5.1 with the optimal equilibrium usage in Corollary 5.3, we observe that analytically it is possible to coordinate the water usage system by achieving the centralized solution in the decentralized problem. Namely, if the optimal solution in both problems is found to be one of the following solutions: $(u_1^{g*}, u_1^{s*}) \in \{(x_1^g, w_1^* - x_1^g), (0, w_1^*), (0,0), (0, x_1^s), (x_1^g, 0), (x_1^g, x_1^s)\}$, where w_1^* is as given in Corollaries 5.1 and 5.3 part (i), then the two solutions are coordinated. This nice property of the solution enables the decision maker (social planner) to impose some values on the revenue-cost parameters in order to make users in the decentralized problem behave as if their are managed centrally.

5.4 Illustrative Numerical Examples

We present some numerical examples to find the optimal solution of the decentralized $(P2)$ and the centralized problems $(P4)$. To facilitate comparison between the two solutions, we consider identical users and select the profit function parameters to satisfy the conditions required for each problem. In particular, in both $(P2)$ and $(P4)$, the parameters are chosen to guarantee that $\vec{u}_{i,2}^* = \vec{x}_{i,2}^*$ (Proposition 5.3), the joint concavity properties of both $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$, $i = 1, 2$, in $(P2)$ and of $\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1)$ (Corollary 5.2) in $(P4)$. Below, in both $(P2)$ and $(P4)$, we investigate the effect of varying the discount rate $\beta \in [0, 1]$ on water usage for two distinct values of the transmissivity coefficient α; one represents the infinite transmissivity case with $\alpha = 0.5$ and the other represents a finite transmissivity case with $\alpha = 0.1$. Both transmissivity cases are studied for a time-variant setting of the profit function parameters as well as where we set $x^g_{i,0} = x^g_{i,1} = x^s_{i,0} = x^s_{i,1} = 1$, for $i = 1, 2$.

5.4.1 Impact of Discount Rate on Water Usage with Infinite Aquifer Transmissivity

Herein, we consider the case of time-variant profit function parameters. Namely, we set $(\rho_1, \rho_2) = (1, 2)$, $(a_1, a_2) = (100, 85)$, $(b_1, b_2) = (10, 15)$, $(c_1, c_2) = (20, 10)$, $(d_1, d_2) = (5, 15)$ and $(h_1, h_2) = (5, 10)$. We also set $\alpha = 0.5$, which represents the infinite transmissivity case. We solve $(P2)$ and $(P4)$ for different values of β over the range $\beta \in [0, 1]$. Table 5.1 summarizes the optimal solution for the decentralized $(P2)$ and centralized $(P4)$ problems accompanied with their respective total discounted profits, $\sum_{i=1}^{2} \Gamma^*_{i,1}(\vec{u}_1, \vec{x}_1)$ and $\tilde{\Gamma}^*_1(\vec{u}_1, \vec{x}_1)$, realized by both users over the entire horizon.

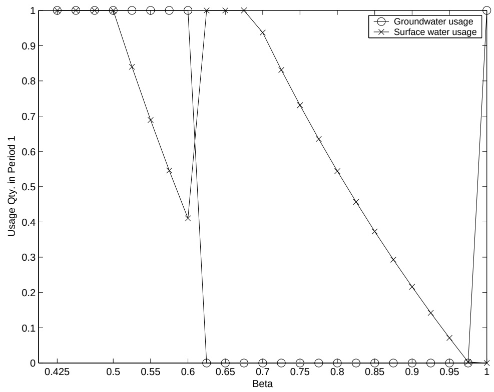
In the decentralized problem, we observe that the groundwater usage in period 1 fluctuates only between the usage two extreme values 0 and 1. More specifically, users consume no groundwater in period 1 for relatively high values of β; for $\beta \in [0.625, 1)$, and prefer to delay the complete usage of their initial stocks to period 2. However, they prefer to fully consume their initial groundwater stocks in period 1 for relatively small values of β; for $\beta \in [0, 0.6]$. This is an expected behavior from users where they prefer to postpone the full usage of their groundwater initial stocks to period 2 as more discounted profits would be achieved in period 2 with high β values, and also to prevent a neighbor from benefiting of lateral flow of groundwater upon pumpage in period 1. Obviously, users behave in an extreme greedy behavior over the entire range of β, where their groundwater usage quantities fluctuate between their extreme values. Under both extremes, non of users experience any lateral transmissivity of groundwater in the aquifer as they pump their initial groundwater stocks fully either in period 1 or in

	Decentralized Solution $(P2)$			Centralized Solution $(P4)$		
β	$\vec{u}^*_{i,1}$	$\vec{u}^*_{i,2}$	$\sum_{i=1}^{2} \Gamma^*_{i,1}(\vec{u}_1, \vec{x}_1)$	$\vec{u}^*_{i,1}$	$\vec{u}^*_{i,2}$	$\tilde{\Gamma}^*_1(\vec{u}_1, \vec{x}_1)$
1.000	$(1, 0)$	$(0, 1)$	470	$(.2, 0)$	$(.8, 1)$	511.6
0.975	$(0, .003)$	$(1, .997)$	497	$(.247, 0)$	$(.753, 1)$	499.5
0.950	$(0, .071)$	$(1, .929)$	484.2	$(.296, 0)$	$(0.704, 1)$	487.5
0.925	$(0, .142)$	$(1, .858)$	471.8	$(.345, 0)$	$(.655, 1)$	475.8
0.900	$(0, .216)$	$(1, .784)$	459.7	$(.396, 0)$	$(.604, 1)$	464.3
0.875	$(0, .293)$	$(1, .707)$	448.1	$(.448, 0)$	$(.552, 1)$	453
0.850	$(0, .373)$	$(1, .627)$	437	$(.5, 0)$	$(.5, 1)$	442
0.825	$(0, .457)$	$(1, .543)$	426.3	$(.554, 0)$	$(.446, 1)$	431.3
0.800	$(0, .544)$	$(1, .456)$	416.1	$(.528, .016)$	$(.472, .984)$	421.1
0.775	$(0, .635)$	$(1, .365)$	406.4	$(.514, .121)$	$(.486, .879)$	411.1
0.750	$(0, .731)$	$(1, .269)$	397.4	$(.5, .231)$	$(.5, .769)$	401.7
0.725	$(0, .831)$	$(1, .169)$	388.9	$(.486, .345)$	$(.514, .655)$	393
0.700	$(0, .937)$	$(1, .063)$	381.1	$(.471, .465)$	$(.529, .535)$	384.9
0.675	$(0, 1)$	$(1, 0)$	374	$(.455, .59)$	$(.545, .41)$	377.5
0.650	$(0, 1)$	$(1, 0)$	367	$(.44, .722)$	$(.56, .278)$	371
0.625	$(0, 1)$	$(1, 0)$	360	$(.423, .86)$	$(.577, .14)$	365.2
0.600	$(1, .41)$	$(0, .59)$	354.7	$(.41, 1)$	$(.59, 0)$	360.4
0.575	$(1, .546)$	$(0, .454)$	350.6	$(.488, 1)$	$(.512, 0)$	356.3
0.550	$(1, .689)$	$(0, .311)$	347.6	$(.571, 1)$	$(.429, 0)$	352.7
0.525	$(1, .84)$	$(0, .16)$	345.7	$(.659, 1)$	$(.341, 0)$	349.8
0.500	$(1, 1)$	$(0, 0)$	345	$(.75, 1)$	$(.25, 0)$	347.5
0.475	$(1, 1)$	$(0, 0)$	345	$(.846, 1)$	$(.154, 0)$	345.9
0.450	$(1, 1)$	$(0, 0)$	345	$(.947, 1)$	$(.053, 0)$	345.1
$(\leq .425)$	$(1, 1)$	$(0, 0)$	345	$(1, 1)$	$(0, 0)$	345

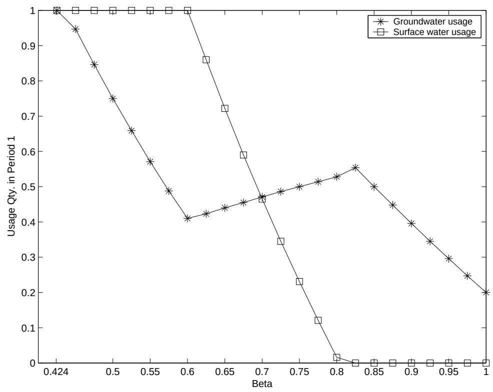
Table 5.1: Effect of β on the optimal solutions of $(P2)$ and $(P4)$ problems for $\alpha = 0.5$, $i = 1, 2$

period 2. However, users behave differently in their surface water usage in period 1. More specifically, their usage quantity in period 1 increases monotonically with β; for $\beta \in [0.625, 1]$, where it decreases suddenly at $\beta = 0.6$ and resumes its increase again until reaching its maximum value; $x^s_1 = 1$, at $\beta = 0.5$ and stays there after that. Their surface water usage responds to a great extent with their extreme groundwater usage. In particular, when they pump no groundwater in period 1 (for high β values), they compensate for that by using as mush surface water as possible as β decreases (specifically for $\beta \in [0.625, 1)$), which is also true for small values of β (specifically for $\beta \in [0, 0.6]$). For $\beta = 1$, it turns out that it is more profitable for users to fully consume their groundwater stock and to use no surface water in period 1. This is attributed to having more discounted pumpage cost if they consume their initial groundwater stocks in period 2, compared to the discounted holding cost of initial surface water stocks if they keep it to period 2. Hence, they prefer the solution $(u_1^{g*}, u_1^{s*}) = (1, 0)$ over the solution $(u_1^{g*}, u_1^{s*}) = (0, 1)$, when $\beta = 1$. Figure 5.2 (a) illustrates the behavior of water usage w.r.t. β for the decentralized problem.

In the centralized solution, we observe that as β decreases over the range $\beta \in [0.825, 1]$, the groundwater usage quantity in period 1 increases monotonically, while it decreases suddenly at $\beta = 0.8$ and keeps decreasing until $\beta = 0.6$, where it resumes its increase again until reaching its maximum value; $x_1^g = 1$, at $\beta = 0.425$. Surface water exhibits a different usage pattern, where users use no surface water along $\beta \in [0.825, 1]$, then they start to increase their usage monotonically at $\beta = 0.8$ until reaching its maximum value; $x_1^s = 1$, at $\beta = 0.6$, and keep using this quantity after that. Since water usage is controlled centrally by the social planner, users do not behave greedily in the groundwater usage, opposite to their greedy behavior in the decentralized problem. In particular, it is more profitable for them to increase their groundwater usage and to use no surface water in period 1 as β decreases over the range $\beta \in [0.825, 1]$. Keeping their full initial stock of surface water to be consumed in period 2 yields the greatest discounted profits among other usage solutions. However, for $\beta \in [0.625, 0.8]$, users exhibit a usage pattern where they partially consume their ground and surface water stocks in period 1. In particular, it turns out that over this range of β, it is more profitable for them to ration their initial ground and surface water stocks to be used across the two periods instead of using them completely in one period only. Again, over this range, groundwater usage in period 1 increases as β decreases. For each β value over the range $\beta \in [0.625, 0.8]$, the decision of how much to use from each source in period 1 is highly dependent on how much cost is realized from such usage. More formally, it turns out that it is more profitable for users to decrease their groundwater usage and to increase their surface water usage in period 1 as β decreases over the range $\beta \in [0.625, 0.8]$. However, for $\beta \in [0.425, 0.6]$, the results show that as β decreases, users realize more profits when they increase their groundwater usage in period 1 up to its maximum value of one and when they fully consume their initial surface water stocks in period 1. After that, it is more profitable for them to completely consume their ground and surface water initial stocks in period 1 as β decreases over the range $\beta \in [0, 0.425]$. Figure 5.2 (b) illustrates the change of surface and groundwater usage w.r.t. β for the centralized problem.



(a) Decentralized Problem



(b) Centralized Problem

Figure 5.2: Water usage in period 1 vs. β: identical users under time-variant setting for $\alpha = 0.5$

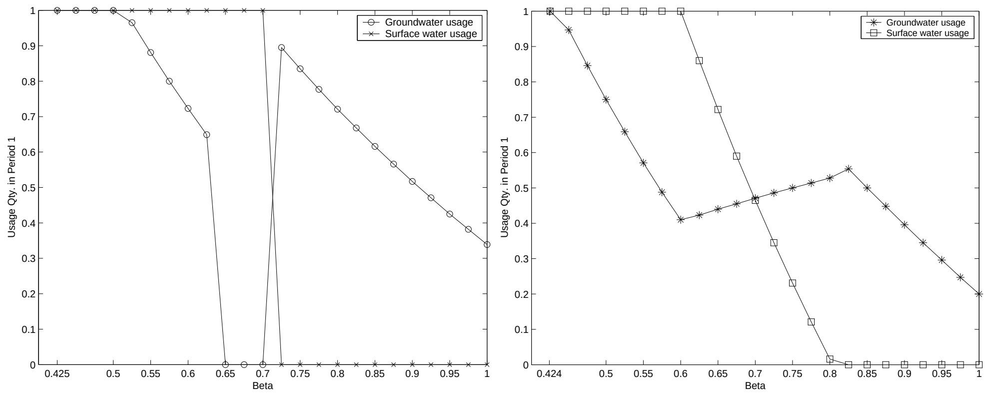
5.4.2 Impact of Discount Rate on Water Usage with Finite Aquifer Transmissivity

We consider the case of time-variant profit function parameters, but under a finite transmissivity case. Namely, we set $\alpha = 0.1$, and keep the values of other parameters as in Example 1. Table 5.2 summarizes the corresponding results for this example. Clearly, the centralized problem gives the same solution as in Example 1 since the solution is independent of α as mentioned previously. In the decentralized problem, users exhibit a more stable and a less greedy behavior in their groundwater pumpage in period 1 compared to their behavior with the infinite transmissivity. More specifically, we observe that they consume their initial groundwater stocks over the two periods and not in a single period as in the infinite transmissivity setting.

	Decentralized Solution $(P2)$			Centralized Solution $(P4)$		
β	$\vec{u}^*_{i,1}$	$\vec{u}^*_{i,2}$	$\sum_{i=1}^{2} \Gamma^*_{i,1}(\vec{u}_1, \vec{x}_1)$	$\vec{u}^*_{i,1}$	$\vec{u}^*_{i,2}$	$\tilde{\Gamma}^*_1(\vec{u}_1, \vec{x}_1)$
1.000	$(.339, 0)$	$(.661, 1)$	509.9	$(.2, 0)$	$(.8, 1)$	511.6
0.975	$(.382, 0)$	$(.618, 1)$	497.8	$(.247, 0)$	$(.753, 1)$	499.5
0.950	$(.425, 0)$	$(.575, 1)$	485.9	$(.296, 0)$	$(.704, 1)$	487.5
0.925	$(.471, 0)$	$(.529, 1)$	474.1	$(.345, 0)$	$(.655, 1)$	475.8
0.900	$(.517, 0)$	$(.483, 1)$	462.7	$(.396, 0)$	$(.604, 1)$	464.3
0.875	$(.566, 0)$	$(.434, 1)$	451.4	$(.448, 0)$	$(.552, 1)$	453
0.850	$(.616, 0)$	$(.384, 1)$	440.5	$(.5, 0)$	$(.5, 1)$	442
0.825	$(.668, 0)$	$(.332, 1)$	429.8	$(.554, 0)$	$(.446, 1)$	431.3
0.800	$(.721, 0)$	$(0.279, 1)$	419.3	$(.528, .016)$	$(.472, .984)$	421.1
0.775	$(.777, 0)$	$(.223, 1)$	409.2	$(.514, .121)$	$(.486, .879)$	411.1
0.750	$(.835, 0)$	$(.165, 1)$	399.4	$(.5, .231)$	$(.5, .769)$	401.7
0.725	$(.895, 0)$	$(.105, 1)$	390	$(.486, .345)$	$(.514, .655)$	393
0.700	$(0, 1)$	$(1, 0)$	381	$(.471, .465)$	$(.529, .535)$	384.9
0.675	$(0, 1)$	$(1, 0)$	374	$(.455, .59)$	$(.545, .41)$	377.5
0.650	$(0, 1)$	$(1, 0)$	367	$(.44, .722)$	$(.56, .278)$	371
0.625	$(.649, 1)$	$(.351, 0)$	360.5	$(.423, .86)$	$(.577, .14)$	365.2
0.600	$(.723, 1)$	$(.277, 0)$	356	$(.41, 1)$	$(.59, 0)$	360.4
0.575	$(.8, 1)$	$(.2, 0)$	352.1	$(.488, 1)$	$(.512, 0)$	356.3
0.550	$(.881, 1)$	$(.119, 0)$	348.7	$(.571, 1)$	$(.429, 0)$	352.7
0.525	$(.965, 1)$	$(.035, 0)$	345.9	$(.659, 1)$	$(.341, 0)$	349.8
0.500	$(1, 1)$	$(0, 0)$	345	$(.75, 1)$	$(.25, 0)$	347.5
0.475	$(1, 1)$	$(0, 0)$	345	$(.846, 1)$	$(.154, 0)$	345.9
0.450	$(1, 1)$	$(0, 0)$	345	$(.947, 1)$	$(.053, 0)$	345.1
$(\leq .425)$	$(1, 1)$	$(0, 0)$	345	$(1, 1)$	$(0, 0)$	345

Table 5.2: Effect of β on the optimal solutions of $(P2)$ and $(P4)$ problems for $\alpha = 0.1$, $i = 1, 2$

We also observe that their groundwater usage in period 1 increases as β decreases over a relatively high range of β; namely, over $\beta \in [0.725, 1]$. Their usage drops almost to zero over the range $\beta \in [0.65, 0.7]$, and after then it resumes its increase as β decreases further until reaching its maximum value at $\beta = 0.425$. Users' groundwater behavior is quite justifiable since with a small finite transmissivity coefficient, lateral flows of groundwater among users would be smaller than those with the infinite transmissivity coefficient. Therefore, users become less conservative about their initial groundwater stocks usage in period 1 and sacrifice some of their stocks through sharing with each other. Obviously, with



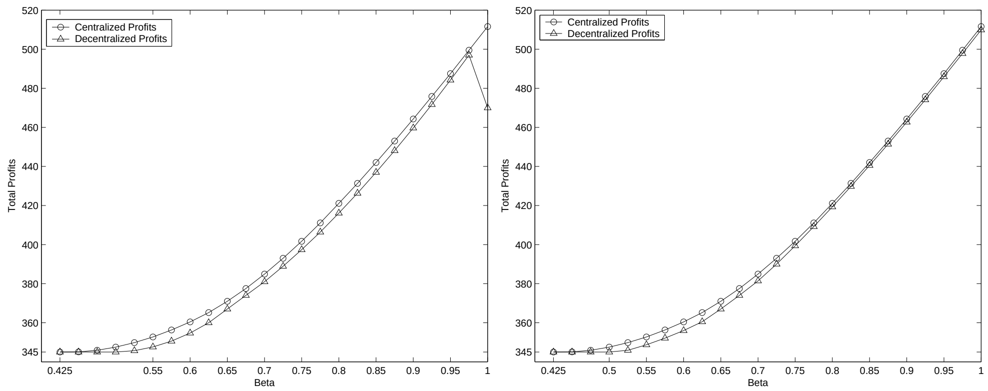
(a) Decentralized Problem

(b) Centralized Problem

Figure 5.3: Water usage in period 1 vs. β: identical users under time-variant setting for $\alpha = 0.1$

high β values, they use less water in period 1 in order to maintain some stock to period 2 and realize more discounted profits. On the other hand, when we look at their behavior in surface water usage, we notice that their usage in period 1 fluctuates between the two extreme values of usage; zero and one, except when $\beta = 0.7$. More specifically, for $\beta \in [0.725, 1]$, they prefer to use no surface water in period 1 and delay the full usage of their initial stocks to period 2 to achieve more discounted profits. However, for $\beta \in [0, 0.625]$, they completely consume their initial surface water stocks in period 1 and keep nothing for period 2 in order to achieve more profits over the entire horizon. Water usage behavior of users in the centralized problem is already interpreted in the previous example. Figure 5.3 illustrates the behavior of water usage w.r.t. β for the decentralized and the centralized problems under the finite transmissivity setting.

In both infinite and finite aquifer transmissivity settings, we observe that both centralized and decentralized profits increase exponentially with β starting at $\beta = 0.425$ for the centralized problem and at $\beta = 0.5$ for the decentralized problem, as shown in Figure 5.4. This implies that the decentralized profits are more sensitive to β (starts to increase at smaller β value) compared to the centralized ones. Moreover, the centralized solution always dominates the decentralized one by realizing more profits over the entire planning horizon. Also, we observe that it is possible for the social planner to achieve (coordinate) the centralized solution under the decentralized problem for $\beta \in [0, 0.425]$.



(a) Infinite transmissivity with $\alpha = 0.5$ (b) Finite transmissivity with $\alpha = 0.1$

Figure 5.4: Total discounted profits vs. β: identical users under time-variant setting

5.5 Summary

In this chapter, we investigated the centralized and decentralized management of conjunctive use of surface and groundwater for a two non-identical users model with time-variant cost-revenue parameters over a time horizon of two periods. In this model surface water is considered a private source of water supply obtained individually by each user from an external supplier while groundwater is common and shared among the two users. The analysis of the decentralized conjunctive water use problem resulted in the characterization of optimal water allocation polices and optimal water usage quantities for each user from each water source in each period. Also, when users are identical, optimal water usage Nash equilibria were determined for each user in each period. Likewise, from the analysis of the centralized conjunctive use problem, we could characterize optimal water allocation policies and optimal water usage quantities for each user from each water source in each period. Again, for identical users' setting, transmissivity coefficient-independent optimal water usage equilibria were characterized for each user from each water source in each period. We also established that, when both users are identical and under careful selection of the model's cost-revenue parameters, it is possible to coordinate the Nash equilibrium with the centralized solution through a single pricing mechanism.

A numerical study with both infinite and finite transmissivity coefficients was conducted. We examined the effect of the discount factor on the water usage behavior of users from each water source under the decentralized and centralized schemes. The results revealed the dominance of the centralized solution over the decentralized one in both infinite and finite transmissivity settings. Also, in both transmissivity settings and under both management problems, the results exhibited that initial surface water stocks are fully consumed in either of the two irrigation periods depending on the discount factor's value. Regarding the groundwater usage, the results showed that it differs across the transmissivity settings and in both settings it is more irregular and erratic in the decentralized problem compared to that in the centralized one.

Chapter 6

Conclusion

In this study, we consider the decentralized and centralized management problems of water resources with multiple users. We first study the decentralized and centralized management problems of groundwater with multiple users under various geometric configurations. Namely, we consider strip, ring, double-layer ring, multi-layer ring and grid configurations. Under each configuration, we investigate two groundwater management problems with multiple users in a dynamic game-theoretic structure over a two-period planning horizon. Under the decentralized management scheme (decentralized problem), each user is allowed to pump groundwater from a common groundwater aquifer making usage decisions individually in a non-cooperative fashion. On the other hand, under the centralized management scheme (centralized problem), users are allowed to pump groundwater from a common aquifer with the supervision of a social planner (water authorities). This work is motivated by the work of Saak and Peterson [52], which considers a model with two identical users sharing a common groundwater aquifer over a two-period planning horizon.

We also investigate the decentralized and centralized conjunctive surface and groundwater use management problems in a dynamic game-theoretic setting over a two-period planning horizon. Under the decentralized use management (decentralized problem), each user is allowed to use surface and groundwater, respectively, from her reservoir and from the common groundwater aquifer making water usage decisions individually in a non-cooperative fashion. Under the centralized use management (centralized problem), users are allowed to use surface and groundwater from their own reservoirs and from the groundwater aquifer, respectively, with the supervision of a social planner.

In this chapter, we provide the contributions of this study as well as some future research directions.

6.1 Contributions

Fresh water sources are essential for sustainability and life on earth. Fresh water supplies mainly come from surface water bodies (rivers, lakes, etc.) or from groundwater aquifers. These sources of fresh water are becoming more limited due to increasing pressures resulting from demographic and/or economic growth and ecological deterioration. Moreover, these sources; especially groundwater, are common and shared by multiple-users the reason which accelerates their depletion over time and increases the chance of having unfair allocation of their quantities among users. The commonality property of groundwater makes underground water laterally to flow within an aquifer in accordance with Darcy's Law, which permits gaming behavior among users upon groundwater pumpage. Therefore, social planner (water authorities) needs to consider more effective water management schemes of water sources to maximize water usage benefits through adopting optimal water usage and allocation polices. Among these management schemes are the decentralized management scheme where users are allowed to make their usage decision in a non-cooperative individual fashion, and the centralized management scheme where water usage decisions are made by the social planner. Furthermore, one of the common practices that water authorities might adopt is to allow conjunctive use of surface and groundwater in order to minimize the undesirable physical, environmental and economical effects of individual source usage and to optimize the water demand/supply balance.

In the first part of this study, we consider groundwater usage when the groundwater stock is shared among multiple users under centralized and decentralized management settings. Our work extends the results of Saak and Peterson [52] to n non-identical users by considering different user geometric configurations, namely, strip, ring, double-layer ring, multi-layer ring and grid configurations, overlying a common groundwater aquifer. It is assumed that transmission of the groundwater is governed by Darcy's Law, which induces a special interaction type among the users between consecutive periods. For a quadratic periodic profit function, general analytical solutions related to the optimal Nash equilibrium usage for the decentralized problem are obtained in each configurations over a course of two periods. Also, we are able to arrive at more compact analytical results for the special case of identical users for the centralized and the decentralized problems. We show that the centralized solution can not be achieved in the decentralized game-theoretic setting via a single pricing mechanism (*i.e.* no coordination). Our analytical results reveal that in strip configuration with identical users, the optimal Nash equilibrium usage quantities oscillate about the optimal Nash equilibrium usage quantities of the ring configuration. We also note that although the optimal solutions of the strip structure do not converge to that of the ring structure as the number of users increase, they are observed to become very close in our numerical examples for the non-extreme users of the strip. In our numerical results of time-invariant setting, we observe that, in both strip and ring configurations in decentralized problems, as the underground water transmission coefficient increases, users become more greedy and use more water. This greedy behavior however adversely affects the system's total discounted profit. We study the effect of changing the crop unit price and yield function parameters on the optimal solution as well as on the realized total profits in the centralized and decentralized problems. In all settings (variant and invariant), the centralized solutions always dominate the decentralized ones by achieving more profits.

In the presence of a salvage function for leftover water stock at the end of problem horizon, we observe that, in both configurations, the centralized solution dominates the decentralized one by realizing more profits from water usage.

Also, in the strip configuration, water usage fluctuates from ends toward the midpoints of the strip. Additionally, in both configurations and in both problems, users allocate part of their available water stocks in the second period to satisfy demands other than the irrigation ones through selling it out according to a given quadratic salvage value function. Under this setting, the policy makers (users and social planner) have more flexibility in allocating their water stock in the second period among two different sources of water demand.

We also consider the groundwater management problems for a double-layer ring configuration where each layer (ring) consists of n identical users with time-variant profit function setting for a two-period planning horizon. We obtain the Nash equilibrium water usage quantities for each user in each layer under the decentralized problem. The equilibrium water usage quantities for each user in each layer are also obtained under the centralized problem. More compact solutions could be obtained for both management problems when all users in both layers are identical. Also, it is found that the solutions of both the decentralized and centralized problems corresponding to the multi-layer ring configuration have the same structure as those in the strip configuration but with non-identical users. Our numerical results also reveal the dominance of the centralized solutions over the decentralized ones.

Lastly, we consider the groundwater management problems of a more general and complex geometric configuration, which is the grid configuration. Although we consider the analysis of the simplest case of square odd and even grids with identical users, unfortunately, we are not able to write the FOC corresponding to the decentralized and centralized problems in nice forms like those in the previous configurations. However, through our conducted extensive numerical study, we could derive two conjectures regarding the number of different solutions of the FOC in the decentralized problem and the solution of the centralized problem. More specifically, we derive the number of different solutions of the FOC of odd and even grids as well as we find that the centralized solution is similar to that in the strip and ring configurations with identical users. Moreover, our numerical results show the symmetry of solutions of users within the same category on the grid; corner, edge or internal users. Also, the centralized solutions dominate the decentralized ones through realizing more discounted profits of water usage over the two periods time horizon.

In the second part of the study, we consider the conjunctive use of surface and groundwater when the groundwater source is shared among two non-identical users under decentralized and centralized management schemes. Each user has her own surface water reservoir, shares a common groundwater aquifer with the other user and uses surface and groundwater stocks conjunctively in a dynamic game-theoretic setting over a planning horizon of two periods. For a quadratic periodic profit function and under well-selected time-variant parameters, general analytical solutions related to the optimal water allocation policies and the optimal Nash equilibria of water usage are obtained under the decentralized management scheme. In addition, general analytical results of the optimal water allocation policies and the optimal equilibrium water usage quantities are obtained under the centralized management scheme.

Our numerical study investigates the effect of the discount rate, on the optimal solution for both problems for identical users having the same, but time-variant parameters, with finite and infinite transmissivity coefficients. We observe that the centralized solutions always dominate the decentralized ones provided that the model parameters are well-chosen to satisfy some structural properties in both problems. We also observe that under certain parameter settings, it is possible to coordinate the conjunctive use system by achieving the centralized solution in the decentralized problem. In addition, total decentralized and centralized profits turn out to increase exponentially with the discount rate.

Our findings fit within the broader literature on management and operating policy making for usage of limited natural resources. We hope that, in the context of water management, our results will aid the decision makers in developing and adopting control policies for more effective and fair usage of water resources.

6.2 Future Research Directions

The main objective of this study in its two parts was to investigate and compare the decentralized and centralized management schemes of water resources under a dynamic game-theoretic setting with multiple users over a planning horizon of two periods. In this section, we present possible research extensions.

In the first part of this study, we conduct all our analysis in all geometric configurations over a planning horizon of two periods. As we mentioned before, Saak and Peterson [52] argue that in a multi-period setting, information about the transmissivity of the groundwater aquifer affects both the the speed of water pumpage as well as the useable life of the aquifer. More specifically, they point out that the lifetime of the aquifer may increase or decrease when users are better informed about the region's hydrology depending on their water usage benefit functions and discount factors. Recall that the decentralized problems corresponding to all configurations we have considered so far are formulated without taking into consideration the information about the aquifer's transmissivity. However, without loss of generality, as we discussed before, the information about the aquifer's transmissivity considered by Saak and Peterson [52] could be easily augmented in our analytical results through considering the transmissivity coefficient, α, as a random variable by taking its expectation in the respective results. In the sequel, the reason for having the two-period setting as adopted in Saak and Peterson [52] still holds in our analysis as well. Nevertheless, we tried to extend the time horizon for more than two periods, but, we found that analysis became more tedious and no analytical results could be obtained. Instead of extending the planning horizon, we considered the time-variant setting of the periodic profit function, where the cost-revenue parameters are taken to vary with time. Under this setting, the two periods are not necessary to be of equal length and this is reflected in the the cost-revenue parameters of their respective periodic profit functions. More specifically, the first period might represent one season having cost-revenue parameters different than those of the second period which might represent another season of different length. The cost-revenue parameters change across the two periods according to water usage cost and to the yielding of grown and irrigated crops. Furthermore, we consider the addition of a salvage function which may be viewed as a proxy for the impact of extending the problem horizon. We consider the salvage function model in the strip and ring configurations where we leave the addition of the salvage function in the other configurations; double-layer ring, multi-layer ring and grid configurations as a future research.

Another future research direction might be the investigation of the centralized and decentralized management schemes for other renewable resources such as fisheries. Fisheries management problems has many common characteristics of those in groundwater management problems. In particular, fisheries in the ecosystem are a common source of fish stock (common property) shared among multi-users. Saak and Peterson [52] discuss that their results of two identical users could be extended to fisheries. Specifically, they consider the simplest setting in which the rate of growth does not depend on the fish population and the fish population dispersal is proportional to the difference in biomasses (populations) across the two users locations. They write the discrete time equation of motion between the two locations (patches) which is quite similar to that of lateral flows of groundwater in the aquifer. Many studies have been devoted to the spatial management of fisheries under dynamic game-theoretic settings. Among these studies is a study by Grøbæk [27] which is literature survey on the fishery economics and game theory. The author describes building game-theoretic models for different types of fishery models. Also, Kvamsdal and Groves [33] analyzed spatial management of a fishery under parameter uncertainty for two fishing areas where intrinsic growth rate is treated as uncertain parameter. One more work by Sanchirico and Wilen [55] investigates the characteristics of an optimally managed spatially explicit renewable resource system. All these works, except Saak and Peterson [52], consider the fisheries spacial management problems dynamically but under a continuous time horizon. One research direction might the analysis of the spacial management of fisheries centrally and decentrally with multiple users over a discrete time horizon of two periods.

In the second part of this study, we started with the simplest case of conjunctive water usage with two non-identical users. Again, the analysis is conducted for a planning horizon of two periods. One line of research extension of this model is considering the analysis of a setting with multi-users, each having her own stock of surface water (in her reservoir) and shares with others a common stock of groundwater in an underground aquifer. Under this setting, different geometric configurations of users, like those considered in the first part of this study, might be a future research extension to focus on. We think that starting with multi-identical users might lower the complexity of the analysis and facilitate the analysis in order to obtain meaningful solutions. In both parts of this study, all cost-revenue parameters as well as aquifer recharge parameters are assumed to be deterministic. One extension might be the inclusion of stochasticity of these parameters in the model (in the first part of the study) and the inclusion of stochastic surface water inflows in the second part of the study.

Chapter 7

Appendix

Proof of Lemma 3.1

(i) We have $\partial g_{i,t}(u_{i,t}, x_{i,t})/\partial u_{i,t} = (\rho_{i,t}a_{i,t} - c_{i,t}x_{i,0} - \rho_{i,t}b_{i,t}u_{i,t}) + c_{i,t}(x_{i,t} - u_{i,t})$. Since $(\rho_{i,t}b_{i,t} + c_{i,t})x_{i,0} < \rho_{i,t}a_{i,t} < (2\rho_{i,t}b_{i,t} + c_{i,t})x_{i,0}$, the result follows.

(ii) The result follows from $\partial^2 g_{i,t}(u_{i,t}, x_{i,t})/\partial(u_{i,t})^2 = -(\rho_{i,t}b_{i,t} + c_{i,t}) < 0$. $\square$

Proof of Proposition 3.1

(i) First, consider the extreme users $i = 1, n$. For $(i,j) \in \{(1,2),(n,n-1)\}$,

$\frac{\partial}{\partial u_{i,1}}\Gamma_{i,1} \ |_{(u_{i,1}=0)} = \rho_{i,1}a_{i,1} - \beta(1-\alpha)(\rho_{i,2}a_{i,2} + c_{i,2}w_1) + \beta\alpha(1-\alpha)c_{i,2}u_{j,1} + \beta(1-\alpha)\rho_{i,2}b_{i,2}(x_1 + w_1 - \alpha u_{j,1})$. Since $0 \le u_{j,1} \le x_1$, the derivative is strictly positive if $\rho_{i,1}a_{i,1} \ge \beta(\rho_{i,2}a_{i,2} + c_{i,2}w_1)$. Similarly, for non-extreme users $i = 2, \cdots, n-1$, $\frac{\partial}{\partial u_{i,1}}\Gamma_{i,1} \ |_{(u_{i,1}=0)} = \rho_{i,1}a_{i,1} - \beta(1-2\alpha)(\rho_{i,2}a_{i,2} + c_{i,2}w_1) + \beta\alpha(1-2\alpha)c_{i,2}(u_{i-1,1} + u_{i+1,1}) + \beta(1-2\alpha)\rho_{i,2}b_{i,2}(x_1 + w_1 - \alpha(u_{i-1,1} + u_{i+1,1}))$. Again, since $0 \le u_{i-1,1} \le x_1$ and $0 \le u_{i+1,1} \le x_1$, the derivative is strictly positive if $\rho_{i,1}a_{i,1} \ge \beta(\rho_{i,2}a_{i,2} + c_{i,2}w_1)$. Therefore, the objective function of the decentralized problem is strictly increasing at $u_{i,1} = 0$, for all i.

(ii) We proceed in two steps. From Lemma 3.1 (ii), we have concavity of $g_{i,1}(u_{i,1}, x_{i,1})$ with respect to $u_{i,1}$. To establish joint concavity of $g_{i,2}(x_{i,2}, x_{i,2})$

in $\vec{u}_1$, for $i = 1, \cdots, n$, we need to show that the Hessian matrix for $g_{i,2}$ is negative semi-definite (having non positive eigenvalues). The structures of the Hessian matrix for the two extreme users and the non-extreme users are different. Below we describe them separately. First, consider the extreme users $i = 1, n$. Substituting the first part of (3.4) in $g_{i,2}(x_{i,2}, x_{i,2})$ yields $g_{i,2}(x_{i,2}, x_{i,2}) = [\rho_{i,2}a_{i,2} + c_{i,2}w_1 - c_{i,2}(1-\alpha)u_{i,1} - c_{i,2}\alpha u_{j,1}][x_1 + w_1 - (1-\alpha)u_{i,1} - \alpha u_{j,1}] - 0.5(\rho_{i,2}b_{i,2} + c_{i,2})[x_1 + w_1 - (1-\alpha)u_{i,1} - \alpha u_{j,1}]^2$.

Then, the diagonal elements of the Hessian for the extreme users are given by

$$\frac{\partial^2 g_{i,2}(x_{i,2}, x_{i,2})}{\partial (u_{k,1})^2} = \begin{cases} (1-\alpha)^2(c_{i,2} - \rho_{i,2}b_{i,2}), & k = i \\ \alpha^2(c_{i,2} - \rho_{i,2}b_{i,2}), & k = j \\ 0, & o.w. \end{cases} \tag{7.1}$$

and the off-diagonal elements are described as

$$\frac{\partial^2 g_{i,2}(x_{i,2}, x_{i,2})}{\partial u_{k,1}\partial u_{l,1}} = \begin{cases} \alpha(1-\alpha)(c_{i,2} - \rho_{i,2}b_{i,2}), & (k,l) = \{(i,j),(j,i)\} \\ 0, & o.w. \end{cases} \tag{7.2}$$

The solution of the characteristic equation corresponding to the above Hessian results in $(n-1)$ zero eigenvalues and the one eigenvalue given by $\lambda = [\alpha^2 + (1-\alpha)^2](c_{i,2} - \rho_{i,2}b_{i,2})$. Hence, $g_{i,2}(x_{i,2}, x_{i,2})$ is jointly concave in $\vec{u}_1$ if and only if $c_{i,2} \leq \rho_{i,2}b_{i,2}$. Next, consider the non-extreme users, $i = 2, \cdots, n-1$. Substituting the second part of (3.4) in $g_{i,2}(x_{i,2}, x_{i,2})$ results in $g_{i,2}(x_{i,2}, x_{i,2}) = [\rho_{i,2}a_{i,2} + c_{i,2}w_1 - c_{i,2}(1-2\alpha)u_{i,1} - c_{i,2}\alpha(u_{i-1,1} + u_{i+1,1})][x_1 + w_1 - (1-2\alpha)u_{i,1} - \alpha(u_{i-1,1} + u_{i+1,1})] - 0.5(\rho_{i,2}b_{i,2} + c_{i,2})[x_1 + w_1 - (1-\alpha)u_{i,1} - \alpha(u_{i-1,1} + u_{i+1,1})]^2$.

The diagonal elements of the Hessian are as follows

$$\frac{\partial^2 g_{i,2}(x_{i,2}, x_{i,2})}{\partial (u_{k,1})^2} = \begin{cases} (1-2\alpha)^2(c_{i,2} - \rho_{i,2}b_{i,2}), & k = i \\ \alpha^2(c_{i,2} - \rho_{i,2}b_{i,2}), & k = i-1, i+1 \\ 0, & o.w. \end{cases} \tag{7.3}$$

and the off-diagonal elements are given as

$$\frac{\partial^2 g_{i,2}(x_{i,2},x_{i,2})}{\partial u_{k,1}\partial u_{l,1}} = \begin{cases} \alpha(1-2\alpha)(c_{i,2}-\rho_{i,2}b_{i,2}), & (k,l)=\{(i-1,i),(i+1,i)\} \\ \alpha(1-2\alpha)(c_{i,2}-\rho_{i,2}b_{i,2}), & (k,l)=\{(i,i-1),(i,i+1)\} \\ \alpha^2(c_{i,2}-\rho_{i,2}b_{i,2}), & (k,l)=(i-1,i+1) \\ \alpha^2(c_{i,2}-\rho_{i,2}b_{i,2}), & (k,l)=(i+1,i-1) \\ 0, & o.w. \end{cases} \tag{7.4}$$

Again, the solution of the characteristic equation corresponding to the above Hessian yields $(n-1)$ zero eigenvalues and one eigenvalue given by $\lambda = [2\alpha^2 + (1-2\alpha)^2](c_{i,2}-\rho_{i,2}b_{i,2})$. Thus, $g_{i,2}(x_{i,2},x_{i,2})$ is jointly concave in $\vec{u}_1$ if and only if $c_{i,2} \leq \rho_{i,2}b_{i,2}$. Since $g_{i,1}(u_{i,1},x_{i,1})$ and $g_{i,2}(x_{i,2},x_{i,2})$ are jointly concave in $\vec{u}_1$, their sum is jointly concave as well. Therefore, the objective function of the decentralized problem is jointly concave in $\vec{u}_1$. $\square$

To facilitate the proofs in Propositions 3.3, 3.4, 3.7 and 3.8, define

$z_{i,2}(y) = \frac{\partial}{\partial u_{i,2}} g_{i,2}(u_{i,2},x_{i,2})|_{(u_{i,2}=x_{i,2}=y)} + \frac{\partial}{\partial x_{i,2}} g_{i,2}(u_{i,2},x_{i,2})|_{(u_{i,2}=x_{i,2}=y)}$

as the sum of partial derivatives of $g_{i,2}(u_{i,2},x_{i,2})$ with respect to $u_{i,2}$ and $x_{i,2}$, respectively, evaluated at $u_{i,2}=x_{i,2}=y$.

Proof of Proposition 3.3

Construction of A and W

A and W are obtained from the first order conditions (FOC) of the objective function in Eqn (3.6) given by

$\frac{\partial}{\partial u_{i,1}} g_{i,1}(u_{i,1},x_{i,1}) - \beta(1-\alpha)z_{i,2}(x_{i,2}) = 0, \ i = 1, n$

$\frac{\partial}{\partial u_{i,1}} g_{i,1}(u_{i,1},x_{i,1}) - \beta(1-2\alpha)z_{i,2}(x_{i,2}) = 0, \ i = 2, \cdots, n-1$

(i) The FOC can be written as $A\vec{u}_1^{**} = W$, from which we have the following

$$
\begin{array}{ll}
e_{(1,1)}u_{1,1}^{**} + e_{(1,2)}u_{2,1}^{**} & = \lambda_1 \\
\sum_{j=0}^{2} e_{(k+1,k+2-j)}u_{k+2-j,1}^{**} & = \lambda_{k+1},\ k = 1, \cdots, n-2 \\
e_{(n,n-1)}u_{n-1,1}^{**} + e_{(n,n)}u_{n,1}^{**} & = \lambda_n
\end{array} \tag{7.5}
$$

where $e_{(i,i)} = \gamma_i$, $i = 1, \cdots, n$, and

$$
e_{(i,j)} = \begin{cases} \sigma_i, & (i,j) = (i,i+1),\ i = 1, \cdots, n-1 \\ \epsilon_i, & (i,j) = (i,i-1),\ i = 2, \cdots, n \\ 0, & o.w. \end{cases}.
$$

From Eqn (7.5), for $k = 1$, we can write $u_{3,1}^{**}$ in terms of $u_{1,1}^{**}$ and $u_{2,1}^{**}$ as $u_{3,1}^{**} = \hat{\lambda}_3 + \hat{e}_{(3,1)}u_{1,1}^{**} + \hat{e}_{(3,2)}u_{2,1}^{**}$, where $\hat{\lambda}_3 = \frac{\lambda_2}{e_{(2,3)}}$ and $\hat{e}_{(3,m)} = -\frac{e_{(2,m)}}{e_{(2,3)}}$, $m = 1, 2$. Likewise, for $k = 2$, we have $u_{4,1}^{**} = \hat{\lambda}_4 + \hat{e}_{(4,1)}u_{1,1}^{**} + \hat{e}_{(4,2)}u_{2,1}^{**}$, where $\hat{\lambda}_4 = \frac{\lambda_3}{e_{(3,4)}} - \frac{e_{(3,3)}}{e_{(3,4)}}\hat{\lambda}_3$, $\hat{e}_{(4,1)} = -\frac{e_{(3,3)}}{e_{(3,4)}}$ and $\hat{e}_{(4,2)} = -[\frac{e_{(3,2)}}{e_{(3,4)}} + \frac{e_{(3,3)}}{e_{(3,4)}}\hat{e}_{(3,2)}]$. For $k = 1, \cdots, n-2$, we have $u_{k+2,1}^{**} = \hat{\lambda}_{k+1} + \hat{e}_{(k+2,1)}u_{1,1}^{**} + \hat{e}_{(k+2,2)}u_{2,1}^{**}$, where $\hat{\lambda}_{k+2}$ and $\hat{e}_{(k+2,m)}$ are as defined. From $u_{k+2,1}^{**}$, we write $u_{n-1,1}^{**}$ and $u_{n,1}^{**}$ as functions of $u_{1,1}^{**}$ and $u_{2,1}^{**}$, for $k = n-3$ and $k = n-2$, respectively. Substituting their respective formulae in Eqn (7.5) gives

$$
\sum_{j=0}^{1} e_{(n,n-j)}\hat{e}_{(n-j,1)}u_{1,1}^{**} + \sum_{j=0}^{1} e_{(n,n-j)}\hat{e}_{(n-j,2)}u_{2,1}^{**} = \lambda_n - \sum_{j=0}^{1} e_{(n,n-j)}\hat{\lambda}_{n-j} \tag{7.6}
$$

Solving the expression with $u_{1,1}^{**}$ and $u_{2,1}^{**}$ in Eqn (7.5) simultaneously with Eqn (7.6) gives the unique solution for $u_{i,1}^{**}$, $i = 1, 2$ stated in the result; and from this, we obtain $u_{k+2,1}^{**}$, for $k = 1, \cdots, n-2$.

(ii) Immediately follows from (i). □

Proof of Corollary 3.2

The given solution is based on solving the system of equations for the KKT conditions as second-order difference equations, in which the variables are defined as the spatial coordinates (location or lexicographic indexes) of the users on the strip. Suppose n is even, then $k = n/2$. From $A\vec{u}_1^{**} = W$, we have the following:

(i) $\gamma u^{**}_{1,1} + \omega u^{**}_{2,1} = \eta$, (ii) $\sigma u^{**}_{i,1} + \epsilon u^{**}_{i+1,1} + \sigma u^{**}_{i+2,1} = \lambda$, for $1 \leq i \leq k-2$, and (iii) $\sigma u^{**}_{k-1,1} + \epsilon u^{**}_{k,1} + \sigma u^{**}_{k+1,1} = \lambda$. We observe that A and W are symmetric around k. That is, users on the left side of the strip ($1 \leq i \leq k$) have the same difference equations correspondingly with those on the right side ($n \geq i \geq k+1$). Therefore, we have $u^{**}_{i,1} = u^{**}_{n-(i+1),1}$, for $1 \leq i \leq n$. In the sequel, (iii) becomes $\sigma u^{**}_{k-1,1} + (\epsilon + \sigma)u^{**}_{k,1} = \lambda$ since $u^{**}_{k,1} = u^{**}_{k+1,1}$. From Elaydi [19] (p. 91-94), (ii) has the general form of a second order difference equation. The assumed cost structure dictates that $\epsilon \leq 2\sigma$ is not feasible and, hence, the solution to the difference equations is solely of the form r^i. Now, suppose that $u^{**}_{i,1} = h_0 + h_1 r^i$, for $i = 1, ..., k$. Then, from (ii), we have (iv) $\sigma(h_0 + h_1 r^i) + \epsilon(h_0 + h_1 r^i r) + \sigma(h_0 + h_1 r^i r^2) - \lambda = 0$. From (iv), we get $h_0 = \lambda/(2\sigma + \epsilon)$ and $(\sigma + \epsilon r + \sigma r^2)h_1 r^i = 0$, which implies, for $h_1, r \neq 0$, $\sigma + \epsilon r + \sigma r^2 = 0$. The last equation is the characteristic equation which has two distinct real roots given by $r_1 = (-\epsilon - \sqrt{\epsilon^2 - 4\sigma^2})/2\sigma$ and $r_2 = (-\epsilon + \sqrt{\epsilon^2 - 4\sigma^2})/2\sigma$, where $|r_1| < |r_2|$. Consequently, $u^{**}_{i,1} = h_0 + h_1(r_1)^i + h_2(r_2)^i$, for $i = 1, ..., k$. Equations (i) and (iii) are indeed the boundary conditions of the difference equations needed to determine h_1 and h_2. Consider user k, from (iii), we have $\sigma(h_0 + h_1(r_1)^{k-1} + h_2(r_2)^{k-1}) + (\sigma + \epsilon)(h_0 + h_1(r_1)^k + h_2(r_2)^k) - \lambda = 0$, from which we can express h_2 as a function of h_1. Now, consider user 1, from (i), we have $\gamma(h_0 + h_1 r_1 + h_2 r_2) + \omega(h_0 + h_1(r_1)^2 + h_2(r_2)^2) - \eta = 0$, from which when solving for h_1 and using the expression of h_2, we get the formula of h_1.

Similarly, suppose n is odd, then $k = (n+1)/2$. From $A\vec{u}^{**}_1 = W$, we have (i), (ii) as in the even case and (iii$'$) $\sigma u^{**}_{k-1,1} + \epsilon u^{**}_{k,1} + \sigma u^{**}_{k+1,1} = 2\sigma u^{**}_{k-1,1} + \epsilon u^{**}_{k,1} = \lambda$, since $u^{**}_{k-1,1} = u^{**}_{k+1,1}$ by symmetry. h_0, r_1 and r_2 are the same as in the even case. Likewise, from (iii$'$) we find the formula of h_2 as function of h_1 and from (i), when solving for h_1 and using the expression of h_2, we get the formula of h_1. Due you to symmetry in both even and odd cases, $u^{**}_{i,1} = u^{**}_{n-i+1,1}$, for $i = 1, ..., k$. The proposed solution is the unique one due to Proposition 3.3 (i). □

Proof of Proposition 3.4

(i) The two-period centralized problem has the following FOC:

$\frac{\partial}{\partial u_{i,1}} g_{i,1}(u_{i,1}, x_{i,1}) - \beta(1-\alpha)z_{i,2}(x_{i,2}) - \beta\alpha z_{j,2}(x_{j,2}) = 0,\ (i,j) \in \{(1,2),(n,n-1)\}$

$\frac{\partial}{\partial u_{i,1}} g_{i,1}(u_{i,1}, x_{i,1}) - \beta(1-2\alpha) z_{i,2}(x_{i,2}) - \beta\alpha[z_{i-1,2}(x_{i-1,2}) + z_{i+1,2}(x_{i+1,2})] = 0,\ i = 2, \cdots, n-1$

which result in

$$
\begin{aligned}
& e_{(1,1)} u^{**}_{1,1} + e_{(1,2)} u^{**}_{2,1} + e_{(1,3)} u^{**}_{3,1} && = \theta_1 \\
& e_{(2,1)} u^{**}_{1,1} + e_{(2,2)} u^{**}_{2,1} + e_{(2,3)} u^{**}_{3,1} + e_{(2,4)} u^{**}_{4,1} && = \theta_2 \\
& \textstyle\sum_{j=0}^{4} e_{(k,k+2-j)} u^{**}_{k+2-j,1} && = \theta_k \\
& e_{(n-1,n-3)} u^{**}_{n-3,1} + e_{(n-1,n-2)} u^{**}_{n-2,1} + e_{(n-1,n-1)} u^{**}_{n-1,1} + e_{(n-1,n)} u^{**}_{n,1} && = \theta_{n-1} \\
& e_{(n,n-2)} u^{**}_{n-2,1} + e_{(n,n-1)} u^{**}_{n-1,1} + e_{(n,n)} u^{**}_{n,1} && = \theta_n
\end{aligned}
\tag{7.7}
$$

where θ_k in the third part of Eqn (7.7) is for $k = 3, \cdots, n-2$, $e_{(i,j)}$ and θ_i are as defined in the result for $i, j = 1, \cdots, n$. Similar to the proof of Proposition 3.3, we get

$$
\sum_{j=0}^{3} e_{(n-1,n-j)} \hat{e}_{(n-j,1)} u^{**}_{1,1} + \sum_{j=0}^{3} e_{(n-1,n-j)} \hat{e}_{(n-j,2)} u^{**}_{2,1} = \theta_{n-1} - \sum_{j=0}^{3} e_{(n-1,n-j)} \hat{\theta}_{n-j}
\tag{7.8}
$$

$$
\sum_{j=0}^{2} e_{(n,n-j)} \hat{e}_{(n-j,1)} u^{**}_{1,1} + \sum_{j=0}^{2} e_{(n,n-j)} \hat{e}_{(n-j,2)} u^{**}_{2,1} = \theta_n - \sum_{j=0}^{2} e_{(n,n-j)} \hat{\theta}_{n-j}
\tag{7.9}
$$

Solving Eqn (7.8) and Eqn (7.9) simultaneously gives the unique solution in the result.

(ii) Immediately follows from (i). □

Proof of Corollary 3.4

For identical users, the FOC are written in a matrix form $\tilde{A}\vec{u}^{**}_1 = \tilde{W}$, where

$$\tilde{A}_{n\times n} = \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & 0 & 0 & 0 & \dots & 0 \\ \phi_2 & \omega_1 & \omega_2 & \phi_3 & 0 & 0 & \dots & 0 \\ \phi_3 & \omega_2 & \omega_1 & \omega_2 & \phi_3 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \phi_3 & \omega_2 & \omega_1 & \omega_2 & \phi_3 \\ 0 & \dots & 0 & 0 & \phi_3 & \omega_2 & \omega_1 & \phi_2 \\ 0 & \dots & 0 & 0 & 0 & \phi_3 & \phi_2 & \phi_1 \end{pmatrix} \text{ and}$$

$\tilde{W}_{n\times 1} = \begin{pmatrix}\theta & \theta & \dots & \theta & \theta\end{pmatrix}^T$, where

$\phi_1 = (\rho_1 b_1 + c_1) + \beta(1-2\alpha+2\alpha^2)(\rho_2 b_2 - c_2)$, $\phi_2 = \beta(2\alpha - 3\alpha^2)(\rho_2 b_2 - c_2)$, $\phi_3 = \beta\alpha^2(\rho_2 b_2 - c_2)$, $\omega_1 = (\rho_1 b_1 + c_1) + \beta(1-4\alpha+6\alpha^2)(\rho_2 b_2 - c_2)$, $\omega_2 = \phi_1 - \omega_1 = 2\beta\alpha(1-2\alpha)(\rho_2 b_2 - c_2)$ and $\theta = \rho_1 a_1 - \beta(\rho_2 a_2 + c_2 w_1) + \beta\rho_2 b_2(x_1 + w_1)$.

(i) Notice that $\tilde{A}$ is a real square matrix. Since $(\rho_t b_t + c_t)x_{t-1} < \rho_t a_t < (2\rho_t b_t + c_t)x_{t-1}$ holds, $\phi_i,\ i = 1,2,3$, θ and $\omega_i,\ i = 1,2$ are non zero, which implies $\tilde{A}$ is of full rank. Therefore, the system $\tilde{A}\vec{u}_1^{**} = \tilde{W}$ has a unique solution. Observe that $\phi_1+\phi_2+\phi_3 = \phi_2+\phi_3+\omega_1+\omega_2 = 2(\phi_3+\omega_2)+\omega_1 = (\rho_1 b_1+c_1)+\beta(\rho_2 b_2-c_2) > 0$ and $\theta = \rho_1 a_1 - \beta(\rho_2 a_2+c_2 w_1)+\beta\rho_2 b_2(x_1+w_1)$. Hence, the unique solution is given by $u_{i,1}^{**} = [\rho_1 a_1 - \beta(\rho_2 a_2 + c_2 w_1) + \beta\rho_2 b_2(x_1 + w_1)]/[(\rho_1 b_1 + c_1) + \beta(\rho_2 b_2 - c_2)] > 0$ and is independent of α.

(ii) Immediately follows from having $u_{i,1}^{**} \le x_1$. $\square$

Proof of Proposition 3.7

Construction of B and Z

B and Z are obtained from the FOC for the two-period decentralized problem given by $\frac{\partial}{\partial u_{i,1}} g_{i,1}(u_{i,1}, x_{i,1}) - \beta(1-2\alpha)z_{i,2}(x_{i,2}) = 0,\ i = 1, \cdots, n$.

(i) The FOC can be written as $B\vec{u}_1^{**} = Z$, from which we have the following

$$\begin{array}{ll} e_{(1,1)}u^{**}_{1,1} + e_{(1,2)}u^{**}_{2,1} + e_{(1,n)}u^{**}_{n,1} & = \lambda_1 \\ \sum_{j=0}^{2} e_{(k+1,k+2-j)}u^{**}_{k+2-j,1} & = \lambda_{k+1}, \ k = 1, \cdots, n-2 \\ e_{(n,1)}u^{**}_{1,1} + e_{(n,n-1)}u^{**}_{n-1,1} + e_{(n,n)}u^{**}_{n,1} & = \lambda_n \end{array} \tag{7.10}$$

where $e_{(i,i)} = \epsilon_i$, $i = 1, \cdots, n$, and

$$e_{(i,j)} = \begin{cases} \sigma_i, & (i,j) \in \{(i,i+1),(1,n)\}, \ i = 1, \cdots, n-1 \\ \sigma_i, & (i,j) \in \{(i,i-1),(n,1)\}, \ i = 2, \cdots, n \\ 0, & o.w. \end{cases}.$$

Similar to Proposition 3.3, from the second formula of Eqn (7.10), for $k = 1, \cdots, n-2$, we have $u^{**}_{k+2,1} = \hat{\lambda}_{k+1} + \hat{e}_{(k+2,1)}u^{**}_{1,1} + \hat{e}_{(k+2,2)}u^{**}_{2,1}$, where $\hat{\lambda}_{k+2}$ and $\hat{e}_{(k+2,m)}$ are as defined before in Proposition 3.3. From $u^{**}_{k+2,1}$, we write $u^{**}_{n-1,1}$ and $u^{**}_{n,1}$ as functions of $u^{**}_{1,1}$ and $u^{**}_{2,1}$, for $k = n-3$ and $k = n-2$, respectively. Substituting their respective formulae in the third equation of Eqn (7.7) gives

$$[e_{(n,1)} + \sum_{j=0}^{1} e_{(n,n-j)}\hat{e}_{(n-j,1)}]u^{**}_{1,1} + \sum_{j=0}^{1} e_{(n,n-j)}\hat{e}_{(n-j,2)}u^{**}_{2,1} = \lambda_n - \sum_{j=0}^{1} e_{(n,n-j)}\hat{\lambda}_{n-j} \tag{7.11}$$

Solving the first equation of Eqn (7.10) simultaneously with Eqn (7.11) gives the unique solution for $u^{**}_{i,1}$, $i = 1, 2$ as given in Proposition 3.7 and from which we find $u^{**}_{k+2,1}$, for $k = 1, \cdots, n-2$.

(ii) Immediately follows from (i). □

Proof of Corollary 3.6

Since λ, σ and ϵ are all negative, B is nonsingular and $B\vec{u}^*_1 = Z$ has a unique solution. The proposed solution clearly satisfies this system of equations. Then, $\lambda > (2\sigma+\epsilon)x_1$ implies $u^*_{i,1} \leq x_1+w_1$ and is the Nash equilibrium. If $\lambda < (2\sigma+\epsilon)x_1$, then $u^*_{i,1} > x_1$. However, for any user i, the maximum groundwater stock available for pumpage in period 1 is x_1 (by assumption). Therefore, since all users must pump the same quantity in period 1 as imposed by the KKT conditions, at equilibrium they optimally pump their maximum stocks of groundwater available in period 1, (*i.e.* $u^{*}_{i,1} = x_1$, $i = 1, \cdots, n$). $\square$

Proof of Proposition 3.8

(i) The two-period centralized problem has the following FOC:

$$\frac{\partial}{\partial u_{i,1}} g_{i,1}(u_{i,1}, x_{i,1}) - \beta(1-2\alpha) z_{i,2}(x_{i,2}) - \beta\alpha[z_{i-1,2}(x_{i-1,2}) + z_{i+1,2}(x_{i+1,2})] = 0$$

for $i = 1, \cdots, n$, which results in

$$\begin{aligned}
&e_{(1,1)}u^{**}_{1,1} + e_{(1,2)}u^{**}_{2,1} + e_{(1,3)}u^{**}_{3,1} + e_{(1,n-1)}u^{**}_{n-1,1} + e_{(1,n)}u^{**}_{n,1} &&= \phi_1\\
&e_{(2,1)}u^{**}_{1,1} + e_{(2,2)}u^{**}_{2,1} + e_{(2,3)}u^{**}_{2,1} + e_{(2,4)}u^{**}_{4,1} + e_{(2,n)}u^{**}_{n,1} &&= \phi_2\\
&\textstyle\sum_{j=0}^{4} e_{(k,k+2-j)}u^{**}_{k+2-j,1} &&= \phi_k\\
&e_{(n-1,1)}u^{**}_{1,1} + e_{(n-1,n-3)}u^{**}_{n-3,1} + e_{(n-1,n-2)}u^{**}_{n-2,1} + e_{(n-1,n-1)}u^{**}_{n-1,1} + e_{(n-1,n)}u^{**}_{n,1} &&= \phi_{n-1}\\
&e_{(n,1)}u^{**}_{1,1} + e_{(n,2)}u^{**}_{2,1} + e_{(n,n-2)}u^{**}_{n-2,1} + e_{(n,n-1)}u^{**}_{n-1,1} + e_{(n,n)}u^{**}_{n,1} &&= \phi_n
\end{aligned} \tag{7.12}$$

where $k = 3, \cdots, n-2$ in the third part of Eqn (7.12), $e_{(i,i)}$ and ϕ_i are as defined in the proposition, for $i = 1, \cdots, n$. For $k = 3$, $u^{**}_{5,1}$ can be written in terms of $u^{**}_{i,1}$, $i = 1, 2, 3, 4$ as $u^{**}_{5,1} = \hat{\phi}_5 + \hat{e}_{(5,1)}u^{**}_{1,1} + \hat{e}_{(5,2)}u^{**}_{2,1} + \hat{e}_{(5,3)}u^{**}_{3,1} + \hat{e}_{(5,4)}u^{**}_{4,1}$, where $\hat{\phi}_5 = \frac{\phi_3}{e_{(3,5)}}$ and $\hat{e}_{(5,m)} = \frac{-e_{(3,m)}}{e_{(3,5)}}$, for $m = 1, 2, 3, 4$. For $k = 3, \cdots, n-2$, we have

$$u^{**}_{k+2,1} = \hat{\phi}_{k+2} + \hat{e}_{(k+2,1)}u^{**}_{1,1} + \hat{e}_{(k+2,2)}u^{**}_{2,1} + \hat{e}_{(k+2,3)}u^{**}_{3,1} + \hat{e}_{(k+2,4)}u^{**}_{4,1} \tag{7.13}$$

where $\hat{\phi}_{k+2}$ and $\hat{e}_{(k+2,m)}$ are as defined in the proposition. After writing $u^{**}_{j,1}$, for $j = n-3, n-2, n-1, n$, as a function of $u^{**}_{i,1}$, $i = 1, 2, 3, 4$ using Eqn (7.13), we substitute their respective formulae in the first two and the last two equations of Eqn (7.12) to obtain

$$\sum_{m=1}^{4}\sum_{i=1}^{4} d_{i,m}u^{**}_{i,1} = \epsilon_m \tag{7.14}$$

where $d_{i,m}$ and ϵ_m are as defined in the proposition, for $i, m = 1, 2, 3, 4$. Eqn (7.14) consists of a 4×4 linear system, where $u^{**}_{i,1}$, $i = 1, 2, 3, 4$, is found by substitution and elimination. Then, $u^{**}_{k+2,1}$, for $k = 3, \cdots, n-2$, is found accordingly using Eqn (7.13).

(ii) Immediately follows from (i). $\square$

Proof of Corollary 3.8

For identical users, the FOC are written in a matrix form $\tilde{B}\vec{u}^{**}_1 = \tilde{W}$, where have

$$\tilde{B}_{n\times n} = \begin{pmatrix} \omega_1 & \omega_2 & \phi_3 & 0 & 0 & 0 & \dots & \phi_3 & \omega_2 \\ \omega_2 & \omega_1 & \omega_2 & \phi_3 & 0 & 0 & \dots & 0 & \phi_3 \\ \phi_3 & \omega_2 & \omega_1 & \omega_2 & \phi_3 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \phi_3 & \omega_2 & \omega_1 & \omega_2 & \phi_3 \\ \phi_3 & 0 & \dots & 0 & 0 & \phi_3 & \omega_2 & \omega_1 & \omega_2 \\ \omega_2 & \phi_3 & \dots & 0 & 0 & 0 & \phi_3 & \omega_2 & \omega_1 \end{pmatrix} \text{ and}$$

$\tilde{W}_{n\times 1} = \begin{pmatrix} \theta & \theta & \dots & \theta & \theta \end{pmatrix}^T$, where ω_i, $i = 1, 2$, and ϕ_i, $i = 1, 2, 3$, are as defined before in Corollary 3.4.

(i) Notice that $\tilde{B}$ is a real square matrix. Since $(\rho_t b_t + c_t)x_{t-1} < \rho_t a_t < (2\rho_t b_t + c_t)x_{t-1})$ holds, $\phi_i,\ i = 1, 2, 3,\ \theta$ and $\omega_i,\ i = 1, 2$ are non zero, which implies $\tilde{A}$ is of full rank. Therefore, the system $\tilde{B}\vec{u}^{**}_1 = \tilde{W}$ has a unique solution. Observe that $2(\phi_3 + \omega_2) + \omega_1 = (\rho_1 b_1 + c_1) + \beta(\rho_2 b_2 - c_2) > 0$ and $\theta = \rho_1 a_1 - \beta(\rho_2 a_2 + c_2 w_1) + \beta\rho_2 b_2(x_1 + w_1)$. Hence, the unique solution is given by $u^{**}_{i,1} = [\rho_1 a_1 - \beta(\rho_2 a_2 + c_2 w_1) + \beta\rho_2 b_2(x_1 + w_1)]/[(\rho_1 b_1 + c_1) + \beta(\rho_2 b_2 - c_2)] > 0$ and is independent of α.

(ii) Similar to Corollary 3.4 (ii). $\square$

Proof of Corollary 3.9

Rests on contradiction. Suppose there exists such a c_t; equating the two solutions, we get $\rho_1 a_1(\rho_2 b_2 - c_2) + (\rho_1 b_1 + c_1)[\rho_2 a_2 + c_2 w_1 - \rho_2 b_2(x_1 + w_1)] = 0^{(\dagger)}$. Since $\rho_2 b_2 \geq c_2$, then $\rho_1 a_1(\rho_2 b_2 - c_2) \geq 0$, and, obviously, $(\rho_1 b_1 + c_1) > 0$. Also, the condition $(\rho_2 b_2 + c_2)(x_1 + w_1) < \rho_2 a_2 < (2\rho_2 b_2 + c_2)(x_1 + w_1)$, implies that $(\rho_2 a_2 - \rho_2 b_2(x_1 + w_1)) > c_2(x_1 + w_1) > 0$. Therefore, the left hand side of Eqn $^{(\dagger)}$ will never be zero a contradiction. Hence, there is no c_t that equates the two solutions for all t. $\square$

Proof of Lemma 3.2

(i) We have $\partial g_{(i,k),t}(u_{(i,k),t}, x_{(i,k),t})/\partial u_{(i,k),t} = (\rho_t a_t - c_t x_{k,0} - \rho_t b_t u_{(i,k),t}) + c_t(x_{(i,k),t} - u_{(i,k),t})$. Since $(\rho_t b_t + c_t)x_{k,0} < \rho_t a_t$ due to Eqn (3.19) the result follows.

(ii) The result follows from $\partial^2 g_{(i,k),t}(u_{(i,k),t}, x_{(i,k),t})/\partial(u_{(i,k),t})^2 = -(\rho_t b_t + c_t) < 0$. $\square$

Proof of Proposition 3.9

(i) For user (i,k), we have $\frac{\partial}{\partial u_{(i,k),1}}\Gamma_{(i,k),1} \mid_{(u_{(i,k),1}=0)} = \rho_1 a_1 - \beta(1-\alpha-2\alpha_k)(\rho_2 a_2 + c_2 w_{k,1} - \alpha c_2[(x_{k,1}+w_{k,1}) - (x_{j,1}+w_{j,1})]) + \beta\alpha_k(1-\alpha-2\alpha_k)c_2(u_{(i-1,k),1} + u_{(i+1,k),1}) + \beta\alpha(1-\alpha-2\alpha_k)c_2 u_{(i,j),1} + \beta(1-\alpha-2\alpha_k)\rho_2 b_2[(1-\alpha)(x_{k,1}+w_{k,1}) + \alpha(x_{j,1}+w_{j,1}) - \alpha_k(u_{(i-1,k),1} + u_{(i+1,k),1}) - \alpha u_{(i,j),1}]$. Since $0 \leq u_{(i-1,k),1} \leq x_{k,1}$, $0 \leq u_{(i+1,k),1} \leq x_{k,1}$ and $0 \leq u_{(i,j),1} \leq x_{j,1}$, $\frac{\partial}{\partial u_{(i,k),1}}\Gamma_{(i,k),1} \mid_{(u_{(i,k),1}=0)} > \rho_1 a_1 - \beta(1-\alpha-2\alpha_k)(\rho_2 a_2 + c_2 w_{k,1} - \alpha c_2[(x_{k,1}+w_{k,1}) - (x_{j,1}+w_{j,1})]) + \beta\alpha_k(1-\alpha-2\alpha_k)c_2(u_{(i-1,k),1} + u_{(i+1,k),1}) + \beta\alpha(1-\alpha-2\alpha_k)c_2 u_{(i,j),1} + \beta(1-\alpha-2\alpha_k)\rho_2 b_2[(1-\alpha-2\alpha_k)x_{k,1} + (1-\alpha)w_{k,1} + \alpha(x_{j,1} + w_{j,1} - u_{(i,j),1})]$. Noting that $(\alpha + 2\alpha_k) \in [0, 0.5]$ implies that derivative is strictly positive if $\rho_1 a_1 \geq \beta(\rho_2 a_2 + c_2 w_{k,1} - \alpha c_2[(x_{k,1}+w_{k,1}) - (x_{j,1}+w_{j,1})])$, $i = 1, \cdots, n,\ j,k = 1,2;\ j \neq k$. Therefore, the objective function of the decentralized problem is strictly increasing at $u_{(i,k),1} = 0$, for $i = 1, \cdots, n,\ k = 1,2,\ t =, 1, 2$.

(ii) We proceed in two steps. From Lemma 3.2 (ii), we have concavity of $g_{(i,k),1}(u_{(i,k),1}, x_{(i,k),1})$ with respect to $u_{(i,k),1}$. To establish joint concavity of

$g_{(i,k),2}(x_{(i,k),2}, x_{(i,k),2})$ in $\vec{u}_1$, for $i = 1, \cdots, n,\ j, k = 1, 2;\ j \neq k$, we need to show that the Hessian matrix for $g_{i,2}(.,.)$ is negative semi-definite (having non positive eigenvalues). Substituting

$x_{(i,k),2} = (1-\alpha)(x_{k,1}+w_{k,1})+\alpha(x_{j,1}+w_{j,1})-(1-\alpha-2\alpha_k)u_{(i,k),1}-\alpha_k[u_{(i-1,k),1}+u_{(i+1,k),1}] - \alpha u_{(i,j),1}$

in $g_{(i,k),2}(x_{(i,k),2}, x_{(i,k),2})$ results in

$g_{(i,k),2}(x_{(i,k),2}, x_{(i,k),2}) = [\rho_2 a_2 - c_2 x_{0,k} + (1-\alpha)c_2(x_{k,1} + w_{k,1}) + \alpha c_2(x_{j,1} + w_{j,1}) - c_2(1-\alpha-2\alpha_k)u_{(i,k),1} - c_2\alpha_k(u_{(i-1,k),1} + u_{(i+1,k),1}) - c_2\alpha u_{(i,j),1}][(1-\alpha)(x_{k,1} + w_{k,1}) + \alpha(x_{j,1} + w_{j,1}) - (1-\alpha-2\alpha_k)u_{(i,k),1} - \alpha_k(u_{(i-1,k),1} + u_{(i+1,k),1}) - \alpha u_{(i,j),1}] - 0.5(\rho_2 b_2+c_2)[(1-\alpha)(x_{k,1}+w_{k,1})+\alpha(x_{j,1}+w_{j,1})-(1-\alpha-2\alpha_k)u_{(i,k),1}-\alpha_k(u_{(i-1,k),1}+u_{(i+1,k),1}) - \alpha u_{(i,j),1}]^2.$

The Hessian matrix corresponding to $g_{(i,k),2}(x_{(i,k),2}, x_{(i,k),2})$ is given below, where the diagonal elements are:

$$\frac{\partial^2 g_{(i,k),2}(.,.)}{\partial (u_{(l,m),1})^2} = \begin{cases} \hat{\alpha}^2 e_2, & (l,m) = (i,k) \\ \alpha_k^2 e_2, & (l,m) \in \{(i-1,k),(i+1,k)\} \\ \alpha^2 e_2, & (l,m) = (i,j) \\ 0, & o.w. \end{cases}$$

and the off-diagonal elements are:

$$\frac{\partial^2 g_{(i,k),2}(.,.)}{\partial u_{(l,m),1}\partial u_{(p,s),1}} = \begin{cases} \alpha_k\hat{\alpha}e_2, & (l,m) = (i,k),\ (p,s) \in \{(i-1,k),(i+1,k)\} \\ \alpha_k\hat{\alpha}e_2, & (l,m) \in \{(i-1,k),(i+1,k)\},\ (p,s) = (i,k) \\ \alpha\hat{\alpha}e_2, & (l,m) = (i,k),\ (p,s) = (i,j) \\ \alpha\hat{\alpha}e_2, & (l,m) = (i,j),\ (p,s) = (i,k) \\ \alpha_k^2 e_2, & (l,m) = (i-1,k),\ (p,s) = (i+1,k) \\ \alpha_k^2 e_2, & (l,m) = (i+1,k),\ (p,s) = (i-1,k) \\ \alpha\alpha_k e_2, & (l,m) = (i-1,k),\ (p,s) = (i,j) \\ \alpha\alpha_k e_2, & (l,m) = (i+1,k),\ (p,s) = (i,j) \\ 0, & o.w. \end{cases}$$

where $\hat{\alpha} = 1-\alpha-2\alpha_k$ and $e_2 = c_2 - \rho_2 b_2$.

The Hessian matrix has $(2n-2)$ zero eigenvalues and two eigenvalues given by $\lambda_1 = 0.5e_2[(\hat{\alpha}^2+\alpha^2+2\alpha_k^2)+\sqrt{(\hat{\alpha}^2+\alpha^2+2\alpha_k^2)^2-8(\alpha_k\hat{\alpha})^2}]$ and

$\lambda_2 = 0.5e_2[(\hat{\alpha}^2+\alpha^2+2\alpha_k^2)-\sqrt{(\hat{\alpha}^2+\alpha^2+2\alpha_k^2)^2-8(\alpha_k\hat{\alpha})^2}]$. The square root value in both λ_1 and λ_2 is non-negative and, hence, it is less that $(\hat{\alpha}^2+\alpha^2+2\alpha_k^2)$ for $(\alpha+2\alpha_k)] \in [0, 0.5]$. Thus, λ_1 and λ_2 are non-positive if and only if $e_2 = c_2-\rho_2 b_2 \leq 0$ or if and only if $c_2 \leq \rho_2 b_2$. Since $g_{(i,k),1}(u_{(i,k),1}, x_{(i,k),1})$ and $g_{(i,k),2}(x_{(i,k),2}, x_{(i,k),2})$ are jointly concave in $\vec{u}_1$, their sum is jointly concave as well. Therefore, the objective function of the decentralized problem is jointly concave in $\vec{u}_1$. $\square$

Proof of Proposition 3.11

By substituting $u^{**}_{(i,k),1} = u^{**}_{k,1}$, for all i and k in $B\vec{u}_1^{**} = Z$, we get the following two equations. Namely, for users in layer 1 (inner layer), we have $2(\epsilon+\gamma)u^{**}_{1,1} + \sigma u^{**}_{2,1} = \lambda_1$ and for those in layer 2 (outer layer), we have $\sigma u^{**}_{1,1} + 2(\epsilon+\gamma)u^{**}_{2,1} = \lambda_2$.

Solving the last two equation simultaneously yields the unique solution given as $u^{**}_{(i,k),1} = \frac{(2\epsilon+\gamma)\lambda_k - \sigma\lambda_j}{(2\epsilon+\gamma)^2-\sigma^2}$, for $i = 1, \cdots, n$, $j, k = 1, 2$, $j \neq k$. Notice that $(2\epsilon+\gamma) = \beta\hat{\alpha}_1(1-\alpha)(c_2-\rho_2 b_2) - (\rho_1 b_1 + c_1) < 0$ and $\sigma = \beta\alpha\hat{\alpha}_1(c_2-\rho_2 b_2) \leq 0$, where $\hat{\alpha}_1 = 1-\alpha-2\alpha_1$. From which, we find that

$(2\epsilon+\gamma)^2 - \sigma^2 = \beta\hat{\alpha}_1^2(1-2\alpha)(c_2-\rho_2 b_2)^2 - 2\beta\hat{\alpha}_1(1-\alpha)(c_2-\rho_2 b_2)(\rho_1 b_1 + c_1) + (\rho_1 b_1 + c_1)^2 > 0$,

which implies that the denominator of $u^{**}_{(i,k),1}$ is strictly positive. For a feasible solution, we need $(2\epsilon+\gamma)\lambda_k - \sigma\lambda_j > 0$. Therefore, the unique solution is optimal if its feasible and it is feasible if $0 < u^{**}_{(i,k),1} \leq x_{k,1}$. Obviously, if $u^{**}_{(i,k),1} \leq x_{k,1}$, then the optimal solution is given by $u^{*}_{(i,k),1} = x_{k,1}$ as users can not use more than the available water stock at the beginning of period 1. $\square$

Proof of Corollary 3.10

The proof is quite straightforward. Since all users in the system have the same initial stocks and the same recharge values, then $\lambda_1 = \lambda_2 = \lambda = \beta\hat{\alpha}_1[\rho_2 a_2 + c_2 w_1 - \rho_2 b_2(x_1+w_1)] - \rho_1 a_1$. Hence, the numerator of $u^{**}_{(i,k),1}$ in Proposition 3.11 becomes $[(2\epsilon+\gamma)-\sigma]\lambda$. By eliminating $[(2\epsilon+\gamma)-\sigma]$ from the numerator and denominator of $u^{**}_{(i,k),1}$, we get the stated unique solution. Since γ and ϵ are negative, $(2\epsilon+\gamma)+\sigma$ is negative as well. However, we can not say anything about the sign of λ. Thus, for $u^{**}_{(i,k),1}$ to be feasible, we need $\lambda < 0$. The given unique solution is optimal if $0 < u^{**}_{(i,k),1} < x_1$, or equivalently if $0 > \lambda > [(2\epsilon+\gamma)+\sigma]x_1$. Similarly, since users can not pump more than their available water stock in period 1, then if $u^{**}_{(i,k),1} > x_1$, the optimal solution is given by $u^{*}_{(i,k),1} = x_1$, for all i and k. $\square$

Proof of Proposition 3.12

By substituting $u^{**}_{(i,k),1} = u^{**}_{k,1}$, for all i and k in $\tilde{B}\vec{u}^{**}_1 = \tilde{Z}$, we get the following two equations. Namely, for users in layer 1 (inner layer), we have $2(\phi+\phi_1+\phi_2)u^{**}_{1,1} + (\eta+2\eta_1)u^{**}_{2,1} = \theta_1$ and for those in layer 2 (outer layer), we have $\sigma u^{**}_{1,1} + 2(\epsilon+\gamma)u^{**}_{2,1} = \theta_2$.

Solving the last two equation simultaneously yields the unique solution given by $u^{**}_{(i,k),1} = \tilde{\theta}/\tilde{\phi} = [(2\phi_2+2\phi_1+\phi)\theta_k - (2\eta_1+\eta)\theta_j]/[(2\phi_2+2\phi_1+\phi)^2 - (2\eta_1+\eta)^2]$, $i = 1, \cdots, n,\ j, k = 1, 2,\ j \neq k$.

One can easily show that $(2\phi_1+\phi_2)+\phi = \beta(1-2\alpha+2\alpha^2-\alpha^2)(c_2-\rho_2 b_2) - (\rho_1 b_1 + c_1)$ and $(2\eta_1+\eta) = 2\beta\alpha(1-\alpha)(c_2-\rho_2 b_2) \leq 0$. From which, we can show that $(2\phi_1+\phi_2)+\phi+(2\eta_1+\eta) = \beta(1-\alpha_1^2)(c_2-\rho_2 b_2) - (\rho_1 b_1 + c_1) < 0$ and $\tilde{\phi} = \beta^2[(1-2\alpha+2\alpha^2-\alpha_1^2) - 4\alpha^2(1-\alpha)^2](c_2-\rho_2 b_2) - 2\beta(1-2\alpha+2\alpha^2-\alpha_1^2)(c_2-\rho_2 b_2)(\rho_1 b_1 + c_1) + (\rho_1 b_1 + c_1)^2 > 0$.

This implies that the denominator of $u^{**}_{(i,k),1}$ is strictly positive. For a feasible solution, we need $\tilde{\theta} > 0$. Therefore, the unique solution is optimal if its feasible and it is feasible if $0 < u^{**}_{(i,k),1} \leq x_{k,1}$. $\square$

Proof of Corollary 3.12

The proof is quite straightforward. Since all users in the system have the same initial stocks and the same recharge values, then $\theta_1 = \theta_2 = \theta = \rho_1 a_1 - \beta[\rho_2 a_2 + c_2 w_1 - \rho_2 b_2(x_1+w_1)]$. Hence, the numerator of $u^{**}_{(i,k),1}$ in Proposition 3.12 becomes $[(2\phi_2+2\phi_1+\phi)-(2\eta_1+\eta)]\theta$. By eliminating $[(2\phi_2+2\phi_1+\phi)-(2\eta_1+\eta)]$ from the numerator and denominator of $u^{**}_{(i,k),1}$, we get the stated unique solution. Recall that in the Proposition 3.12, we showed that $(2\phi_1+\phi_2)+\phi+(2\eta_1+\eta) =$

$\beta(1-\alpha_1^2)(c_2-\rho_2 b_2)-(\rho_1 b_1+c_1)<0$

However, we can not say anything about the sign of θ. Thus, for $u^{**}_{(i,k),1}$ to be feasible, we need $\theta<0$. The given unique solution is optimal if $0<u^{**}_{(i,k),1}<x_1$, or equivalently if $0>\theta>[(2\phi_1+\phi_2)+\phi+(2\eta_1+\eta)]x_1$. Similarly, since users can not pump more than their available water stock in period 1, then if $u^{**}_{(i,k),1}>x_1$, the optimal solution is given by $u^{*}_{(i,k),1}=x_1$, for all i and k. $\square$

Proof of Proposition 3.13

The proof of this proposition resembles to a great extent the proof of Proposition 3.9 of the double-layer configuration.

(i) For user (i,k), we have $\frac{\partial}{\partial u_{(i,k),1}}\Gamma_{(i,k),1}\mid_{(u_{(i,k),1}=0)}=\rho_1 a_1-\beta(1-2\alpha_k-\alpha_{(k,k-1)}-\alpha_{(k,k+1)})(\rho_2 a_2+c_2 w_{k,1}-\alpha_{(k,k-1)}c_2[(x_{k,1}+w_{k,1})-(x_{k-1,1}+w_{k-1,1})]-\alpha_{(k,k+1)}c_2[(x_{k,1}+w_{k,1})-(x_{k+1,1}+w_{k+1,1})]])+\beta\alpha_k(1-2\alpha_k-\alpha_{(k,k-1)}-\alpha_{(k,k+1)})c_2(u_{(i-1,k),1}+u_{(i+1,k),1})+\beta(1-2\alpha_k-\alpha_{(k,k-1)}-\alpha_{(k,k+1)})c_2[\alpha_{(k,k-1)}u_{(i,k-1),1}+\alpha_{(k,k+1)}u_{(i,k+1),1}]+\beta(1-2\alpha_k-\alpha_{(k,k-1)}-\alpha_{(k,k+1)})\rho_2 b_2[(1-\alpha_{(k,k-1)}-\alpha_{(k,k+1)})(x_{k,1}+w_{k,1})+\alpha_{(k,k-1)}(x_{k-1,1}+w_{k-1,1})+\alpha_{(k,k+1)}(x_{k+1,1}+w_{k+1,1})-\alpha_k(u_{(i-1,k),1}+u_{(i+1,k),1})-\alpha_{(k,k-1)}u_{(i,k-1),1}-\alpha_{(k,k+1)}u_{(i,k+1),1}]$. Since $0\le u_{(i-1,k),1}\le x_{k,1}$, $0\le u_{(i+1,k),1}\le x_{k,1}$, $0\le u_{(i,k-1),1}\le x_{k-1,1}$, $0\le u_{(i,k+1),1}\le x_{k+1,1}$ and $(2\alpha_k+\alpha_{k-1}+\alpha_{k+1})\in[0,0.5]$, the derivative is strictly positive if $\rho_1 a_1\ge\beta(\rho_2 a_2+c_2 w_{k,1}-\alpha_{(k,k-1)}c_2[(x_{k,1}+w_{k,1})-(x_{k-1,1}+w_{k-1,1})]-\alpha_{(k,k+1)}c_2[(x_{k,1}+w_{k,1})-(x_{k+1,1}+w_{k+1,1})]])$, $i=1,\cdots,n,\ k=1,\cdots,m$. Therefore, the objective function of the decentralized problem is strictly increasing at $u_{(i,k),1}=0$, for $i=1,\cdots,n,\ k=1,\cdot,m,\ t=,1,2$.

(ii) We proceed in two steps. From Lemma 3.2 (ii), we have concavity of $g_{(i,k),1}(u_{(i,k),1},x_{(i,k),1})$ with respect to $u_{(i,k),1}$. To establish joint concavity of $g_{i,2}(x_{(i,k),2},x_{(i,k),2})$ in $\vec{u}_1$, for $i=1,\cdots,n,\ j,k=1,\cdots,m$, we need to show that the Hessian matrix for $g_{i,2}(.,.)$ is negative semi-definite (having non positive eigenvalues). Substituting

$x_{(i,k),t+1}=(1-\alpha_{(k,k-1)}-\alpha_{(k,k+1)})(x_{k,t}+w_{k,t})+\alpha_{(k,k-1)}(x_{k-1,t}+w_{k-1,t})+\alpha_{(k,k+1)}(x_{k+1,t}+w_{k+1,t})-(1-2\alpha_k-\alpha_{(k,k-1)}-\alpha_{(k,k+1)})u_{(i,k),t}-\alpha_k[u_{(i-1,k),t}+u_{(i+1,k),t}]-\alpha_{(k,k-1)}u_{(i,k-1),t}-\alpha_{(k,k+1)}u_{(i,k+1),t}$

in $g_{(i,k),2}(x_{(i,k),2}, x_{(i,k),2})$ and constructing the Hessian matrix from the resulted function gives that the diagonal elements of the Hessian are as follows

$$\frac{\partial^2 g_{(i,k),2}(.,.)}{\partial (u_{(l,m),1})^2} = \begin{cases} \tilde{\alpha}^2 e_2, & (l,m) = (i,k) \\ \alpha_k^2 e_2, & (l,m) \in \{(i-1,k),(i+1,k)\} \\ \alpha_{(k,k-1)}^2 e_2, & (l,m) = (i,k-1) \\ \alpha_{(k,k+1)}^2 e_2, & (l,m) = (i,k+1) \\ 0, & o.w. \end{cases}$$

and the off-diagonal elements are given as

$$\frac{\partial^2 g_{(i,k),2}(.,.)}{\partial u_{(l,m),1} \partial u_{(p,s),1}} = \begin{cases} \alpha_k \tilde{\alpha} e_2, & (l,m) = (i,k),\ (p,s) = (i-1,k) \\ \alpha_k \tilde{\alpha} e_2, & (l,m) = (i,k),\ (p,s) = (i+1,k) \\ \alpha_k \tilde{\alpha} e_2, & (l,m) = (i-1,k),\ (p,s) = (i,k) \\ \alpha_k \tilde{\alpha} e_2, & (l,m) = (i+1,k),\ (p,s) = (i,k) \\ \alpha_{(k,k-1)} \tilde{\alpha} e_2, & (l,m) = (i,k),\ (p,s) = (i,k-1) \\ \alpha_{(k,k-1)} \tilde{\alpha} e_2, & (l,m) = (i,k-1),\ (p,s) = (i,k) \\ \alpha_{(k,k+1)} \tilde{\alpha} e_2, & (l,m) = (i,k),\ (p,s) = (i,k+1) \\ \alpha_{(k,k+1)} \tilde{\alpha} e_2, & (l,m) = (i,k+1),\ (p,s) = (i,k) \\ \alpha_k^2 e_2, & (l,m) = (i-1,k),\ (p,s) = (i+1,k) \\ \alpha_k^2 e_2, & (l,m) = (i+1,k),\ (p,s) = (i-1,k) \\ \alpha_{(k,k-1)} \alpha_k e_2, & (l,m) = (i-1,k),\ (p,s) = (i,k-1) \\ \alpha_{(k,k-1)} \alpha_k e_2, & (l,m) = (i+1,k),\ (p,s) = (i,k-1) \\ \alpha_{(k,k-1)} \alpha_k e_2, & (l,m) = (i+1,k),\ (p,s) = (i,k-1) \\ \alpha_{(k,k-1)} \alpha_k e_2, & (l,m) = (i,k-1),\ (p,s) = (i+,k) \\ \alpha_{(,k+1)} \alpha_k e_2, & (l,m) = (i-1,k),\ (p,s) = (i,k+1) \\ \alpha_{(k,k+1)} \alpha_k e_2, & (l,m) = (i,k+1),\ (p,s) = (i-1,k) \\ \alpha_{(k,k+1)} \alpha_k e_2, & (l,m) = (i+1,k),\ (p,s) = (i,k+1) \\ \alpha_{(k,k+1)} \alpha_k e_2, & (l,m) = (i,k+1),\ (p,s) = (i+1,k) \\ \alpha_{(k,k-1)} \alpha_{(k,k+1)} e_2, & (l,m) = (i,k-1),\ (p,s) = (i,k+1) \\ \alpha_{(k,k-1)} \alpha_{(k,k+1)} e_2, & (l,m) = (i,k+1),\ (p,s) = (i,k-1) \\ 0, & o.w. \end{cases}$$

where $\tilde{\alpha} = 1 - 2\alpha_k - \alpha_{(k,k-1)} - \alpha_{(k,k+1)}$ and $e_2 = c_2 - \rho_2 b_2$.

The Hessian matrix has at least $(2n-5)$ zero eigenvalues and five eigenvalues given by a fifth-order characteristic polynomial function obtained from solving $det\ (\partial^2 g_{(i,k),2}(.,.)-\lambda I)=0$, where det is the determinant, I is a $2n \times 2n$ identity matrix and λ is the eigenvalue. Unfortunately, it was not easy to come up with a compact form of the characteristic polynomial due to the large size of the Hessian matrix. More specifically, we need to find the determinant of a 5×5 parametric matrix. However, we know that the characteristic polynomial $f(\lambda)$ will have the general form $f(\lambda)=\lambda^5+d_4\lambda^4+d_3\lambda^3+d_2\lambda^2+d_1\lambda^1+d_0$, where d_i will be a function of the revenue-cost parameters (namely, ρ_2, b_2 and c_2) and the transmissivity coefficients ($\tilde{\alpha}$, α_k, α_{k-1} and α_{k+1}), for $i=1,\cdots,5$. Solving for the roots of this polynomial analytically is not an easy task as well. Therefore, numerically, if the parameters of the problem are selected carefully such that the roots of $f(\lambda)$ are non-positive, then the Hessian is negative semi-definite and, hence, $g_{(i,k),2}(x_{(i,k),2},x_{(i,k),2})$ is jointly concave in $\vec{u}_1$. Consequently, the objective function of the decentralized problem is jointly concave in $\vec{u}_1$. $\square$

Proof of Proposition 3.15

(i) Similar to the proof in Proposition 3.1 (ii) and, hence, omitted.

(ii) We proceed in two steps. From Lemma 3.1 (ii), we have concavity of $g_{i,1}(u_{i,1},x_{i,1})$ with respect to $u_{i,1}$. To establish joint concavity of $g_{i,2}(x_{i,2},x_{i,2})$ in $\vec{u}_1$, for $i=1,...,n$, we need to show that the Hessian matrix for $g_{i,2}(x_{i,2},x_{i,2})$ is negative semi-definite (having non positive eigenvalues). The structures of the Hessian matrix for corner, edge and internal users are different. Below we describe them separately. First, consider corner users on the grid. We substitute the first part of Eqn (3.35) in $g_{i,2}(x_{i,2},x_{i,2})$ and, from the first and second derivatives of $g_{i,2}(x_{i,2},x_{i,2})$ with respect to $u_{i,1}\in S_1$, we find the elements of the corresponding Hessian matrix. The diagonal elements of the Hessian matrix are found to be as follows

$$\frac{\partial^2 g_{i,2}(x_{i,2},x_{i,2})}{\partial(u_{l,1})^2}=\begin{cases}(1-2\alpha)^2(c_2-\rho_2 b_2), & l=i\\ \alpha^2(c_2-\rho_2 b_2), & l=j,k\\ 0, & o.w.\end{cases} \tag{7.15}$$

and the off-diagonal elements are given as

$$\frac{\partial^2 g_{i,2}(x_{i,2},x_{i,2})}{\partial u_{l,1}\partial u_{m,1}} = \begin{cases} \alpha(1-2\alpha)(c_2-\rho_2 b_2), & (l,m) = \{(j,i),(k,i),(i,j),(i,k)\} \\ \alpha^2(c_2-\rho_2 b+_2), & (l,m) = \{(j,k),(k,j)\} \\ 0, & o.w. \end{cases} \tag{7.16}$$

The Hessian matrix has $(n-1)$ zero eigenvalues and the remaining is given by $\lambda = [2\alpha^2 + (1-2\alpha)^2](c_2 - \rho_2 b_2)$. Thus, $g_{i,2}(x_{i,2}, x_{i,2})$ is jointly concave in $\vec{u}_1$ if and only if $c_2 \leq \rho_2 b_2$, $i = 1, \cdots, 4$, since each grid has four corner users. Next, we consider edge users on the grid. We substitute the second part of Eqn (3.35) in $g_{i,2}(x_{i,2}, x_{i,2})$ and, from the first and second derivatives of $g_{i,2}(x_{i,2}, x_{i,2})$ with respect to $u_{i,1} \in S_2$, we determine the elements of its Hessian matrix. The diagonal elements of the Hessian matrix are given by

$$\frac{\partial^2 g_{i,2}(x_{i,2},x_{i,2})}{\partial (u_{m,1})^2} = \begin{cases} (1-3\alpha)^2(c_2-\rho_2 b_2), & m = i \\ \alpha^2(c-\rho b), & m = j,k,l \\ 0, & o.w. \end{cases} \tag{7.17}$$

and the off-diagonal elements are given as

$$\frac{\partial^2 g_{i,2}(x_{i,2},x_{i,2})}{\partial u_{m,1}\partial u_{r,1}} = \begin{cases} \alpha(1-3\alpha)(c_2-\rho_2 b_2), & (m,r) = \{(j,i),(k,i),(l,i)\} \\ \alpha(1-3\alpha)(c_2-\rho_2 b_2), & (m,r) = \{(i,j),(i,k),(i,l)\} \\ \alpha^2(c_2-\rho_2 b_2), & (m,r) = \{(j,k),(k,j),(j,l)\} \\ \alpha^2(c_2-\rho_2 b_2), & (m,r) = \{(l,j),(k,l),(l,k)\} \\ 0, & o.w. \end{cases} \tag{7.18}$$

The Hessian matrix has $(n-2)$ zero eigenvalues, one is given by

$\lambda = 0.5[(3\alpha^2 + (1-3\alpha)^2) - \sqrt{(3\alpha^2 + (1-3\alpha)^2)^2 - 4\alpha^2(1-3\alpha)^2}](c_2 - \rho_2 b_2)$ and one is given by

$\tilde{\lambda} = 0.5[(3\alpha^2 + (1-3\alpha)^2) + \sqrt{(3\alpha^2 + (1-3\alpha)^2)^2 - 4\alpha^2(1-3\alpha)^2}](c_2 - \rho_2 b_2)$. Thus, $g_{i,2}(x_{i,2}, x_{i,2})$ is jointly concave in $\vec{u}_1$ if and only if λ, $\tilde{\lambda} \leq 0$ if and only $c_2 \leq \rho_2 b_2$, since $[(3\alpha^2 + (1-3\alpha)^2) \pm \sqrt{(3\alpha^2 + (1-3\alpha)^2)^2 - 4\alpha^2(1-3\alpha)^2}] \geq 0$, for $i = 1, \cdots, n_e$, where n_e is the number of edge users on the grid. Finally, we consider internal users on the grid. Similarly, we substitute the last part of Eqn (3.35) in $g_{i,2}(x_{i,2}, x_{i,2})$ and, from the first and second derivatives of $g_{i,2}(x_{i,2}, x_{i,2})$ with respect to $u_{i,1} \in S_3$, we find the elements of its Hessian matrix. The diagonal elements of the Hessian matrix are as follows

$$\frac{\partial^2 g_{i,2}(x_{i,2}, x_{i,2})}{\partial (u_{r,1})^2} = \begin{cases} (1-4\alpha)^2(c_2 - \rho_2 b_2), & r = i \\ \alpha^2(c_2 - \rho_2 b_2), & r = j, k, l, m \\ 0, & o.w. \end{cases} \tag{7.19}$$

and the off-diagonal elements are given as

$$\frac{\partial^2 g_{i,2}(x_{i,2}, x_{i,2})}{\partial u_{r,1} \partial u_{s,1}} = \begin{cases} \alpha(1-4\alpha)(c_2 - \rho_2 b_2), & (r,s) = \{(j,i), (k,i), (l,i), (m,i)\} \\ \alpha(1-4\alpha)(c_2 - \rho_2 b_2), & (r,s) = \{(i,j), (i,k), (i,l), (i,m)\} \\ \alpha^2(c_2 - \rho_2 b_2), & (r,s) = \{(j,k), (j,l), (j,m)\} \\ \alpha^2(c_2 - \rho_2 b_2), & (r,s) = \{(k,l), (k,m), (l,m)\} \\ \alpha^2(c_2 - \rho_2 b_2), & (r,s) = \{(k,j), (l,j), (m,j)\} \\ \alpha^2(c_2 - \rho_2 b_2), & (r,s) = \{(l,k), (m,k), (m,l)\} \\ 0, & o.w. \end{cases} \tag{7.20}$$

Unlike corner and edge users, the eigenvalues of this Hessian matrix are not easy to be determined. However, when we solve for the eigenvalues, we find that the characteristic function $f(\lambda)$ of this Hessian is a fifth order polynomial, given by $f(\lambda) = \lambda^5 + \gamma_0\lambda^4 + \gamma_1\lambda^3 + \gamma_2\lambda^2 + \gamma_3\lambda + \gamma_4 = 0$, where γ_j, $j = 0, \cdots, 4$ are as defined before in the result. Therefore, $g_{i,2}(x_{i,2}, x_{i,2})$ is jointly concave in $\vec{u}_1$ if and only if $\lambda_k \leq 0$, for all $k = 1, \cdots, 5$, where λ_k's are the roots of the characteristic function $f(\lambda)$, for $i = 1, \cdots, n_i$ and n_i is the number of internal users on the grid. Accordingly, when $g_{i,2}(x_{i,2}, x_{i,2})$ is jointly concave in $\vec{u}_1$, $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$ will be jointly concave in $\vec{u}_1$ as well, since, from Lemma 3.1 (ii), $g_{i,1}(u_{i,1}, 1)$ is concave in $u_{i,1}$. $\square$

Proof of Proposition 3.17

Let us write the solution in Eqn (3.51) as $u^{**}_{i,2} = \theta_0 + \theta_1 x_{i,2}$, where

$\theta_0 = (\rho_{i,2}a_{i,2} - c_{i,2}x_{i,0} - \pi_{i,1})/(\rho_{i,2}b_{i,2} + c_{i,2} + \pi_{i,2})$ and

$\theta_1 = (c_{i,2} + \pi_{i,2})/(\rho_{i,2}b_{i,2} + c_{i,2} + \pi_{i,2})$, $i = 1, \cdots, n$.

We proceed in two steps. From Lemma 3.1 (ii), we have concavity of $g_{i,1}(u_{i,1}, x_{i,1})$ with respect to $u_{i,1}$. To establish joint concavity of $\tilde{g}_{i,2}(u^{**}_{i,2}, x_{i,2})$ in $\vec{u}_1$, for $i = 1, \cdots, n$, we need to show that the Hessian matrix for $\tilde{g}_{i,2}(u^{**}_{i,2}, x_{i,2})$ is negative semi-definite (having non positive eigenvalues). The structures of the Hessian matrix for the two extreme users and the non-extreme users are different. Below we describe them separately. First, consider the extreme users $i = 1, n$.

The diagonal elements of the corresponding Hessian matrix for the extreme users are given by

$$\frac{\partial^2 \tilde{g}_{i,2}(u^{**}_{i,2}, x_{i,2})}{\partial (u_{k,1})^2} = \begin{cases} (1-\alpha)^2\kappa, & k = i \\ \alpha^2\kappa, & k = j \\ 0, & o.w. \end{cases} \tag{7.21}$$

and the off-diagonal elements are described as

$$\frac{\partial^2 \tilde{g}_{i,2}(u^{**}_{i,2}, x_{i,2})}{\partial u_{k,1}\partial u_{l,1}} = \begin{cases} \alpha(1-\alpha)\kappa, & (k,l) = \{(i,j),(j,i)\} \\ 0, & o.w. \end{cases} \tag{7.22}$$

where $\kappa = 2\theta_1 c_{i,2} - \theta_1^2(\rho_{i,2}b_{i,2} + c_{i,2}) - (1-\theta_1)^2\pi_2$.

The Hessian matrix has $(n-1)$ zero eigenvalues and the remaining is given by $\lambda = [\alpha^2 + (1-\alpha)^2]\kappa$. Hence, $\tilde{g}_{i,2}(u^{**}_{i,2}, x_{i,2})$ is jointly concave in $\vec{u}_1$ if and only if $\kappa \le 0$. Substituting the respective values of θ_0 and θ_1 in to κ and doing some algebraic simplifications result in the following equivalent condition

$$2c_{i,2}(c_{i,2}+\pi_{i,2})(\rho_{i,2}b_{i,2}+c_{i,2}+\pi_{i,2}) - (\rho_{i,2}b_{i,2})^2\pi_{i,2} - (c_{i,2}+\pi_{i,2})^2(\rho_{i,2}b_{i,2}+c_{i,2}) \le 0.$$

Next, consider the non-extreme users, $i = 2, \cdots, n-1$. The diagonal elements of the corresponding Hessian matrix are as follows

$$\frac{\partial^2 \tilde{g}_{i,2}(u^{**}_{i,2}, x_{i,2})}{\partial (u_{k,1})^2} = \begin{cases} (1-2\alpha)^2\kappa, & k = i \\ \alpha^2\kappa, & k = i-1, i+1 \\ 0, & o.w. \end{cases} \tag{7.23}$$

and the off-diagonal elements are given as

$$\frac{\partial^2 \tilde{g}_{i,2}(u^{**}_{i,2}, x_{i,2})}{\partial u_{k,1}\partial u_{l,1}} = \begin{cases} \alpha(1-2\alpha)\kappa, & (k,l) = \{(i-1,i),(i+1,i)\} \\ \alpha(1-2\alpha)\kappa, & (k,l) = \{(i,i-1),(i,i+1)\} \\ \alpha^2\kappa, \alpha^2\kappa, & (k,l) = \{(i-1,i+1),(i+1,i-1)\} \\ 0, & o.w. \end{cases} \tag{7.24}$$

Again the Hessian matrix has $(n-1)$ zero eigenvalues and the remaining is given by $\lambda = [2\alpha^2 + (1-2\alpha)^2]\kappa$. Then, the proofs follows similar to that of the extreme user. Thus, $\tilde{g}_{i,2}(u^{**}_{i,2}, x_{i,2})$ is jointly concave in $\vec{u}_1$ if and only if

$$2c_{i,2}(c_{i,2}+\pi_{i,2})(\rho_{i,2}b_{i,2}+c_{i,2}+\pi_{i,2})-(\rho_{i,2}b_{i,2})^2\pi_{i,2}-(c_{i,2}+\pi_{i,2})^2(\rho_{i,2}b_{i,2}+c_{i,2}) \leq 0.$$

Since $g_{i,1}(u_{i,1}, x_{i,1})$ and $\tilde{g}_{i,2}(u^{**}_{i,2}, x_{i,2})$ are jointly concave in $\vec{u}_1$, their sum is jointly concave as well. Therefore, the objective function of the decentralized problem is jointly concave in $\vec{u}_1$. $\square$

Proof of Proposition 5.1

To establish strict joint concavity of $f_{i,t}(\vec{u}_{i,t}, \vec{x}_{i,t})$, we need to show that the Hessian matrix corresponding to $f_{i,t}(\vec{u}_{i,t}, \vec{x}_{i,t})$ is negative semi-definite (*i.e.* having non-positive eigenvalues).

From $f_{i,t}(\vec{u}_{i,t}, \vec{x}_{i,t})$, we have

$\frac{\partial^2 f_{i,t}(.,.)}{\partial (u^g_{i,t})^2} = -(\rho_{i,t}b_{i,t} + 2d_{i,t})$, $\frac{\partial^2 f_{i,t}(.,.)}{\partial u^g_{i,t}\partial u^s_{i,t}} = \frac{\partial^2 f_{i,t}(.,.)}{\partial u^s_{i,t}\partial u^g_{i,t}} = -\rho_{i,t}b_{i,t}$ and $\frac{\partial^2 f_{i,t}(.,.)}{\partial (u^s_{i,t})^2} = -\rho_{i,t}b_{i,t}$, $i = 1, 2,\ t = 1, 2.$

Therefore, the Hessian matrix, $\nabla^2 f_{i,t}(\vec{u}_{i,t}, \vec{x}_{i,t})$, is given by

$$\nabla^2 f_{i,t}(\vec{u}_{i,t}, \vec{x}_{i,t}) = \begin{pmatrix} \frac{\partial^2 f_{i,t}(.,.)}{\partial (u^g_{i,t})^2} & \frac{\partial^2 f_{i,t}(.,.)}{\partial u^g_{i,t} \partial u^s_{i,t}} \\ \frac{\partial^2 f_{i,t}(.,.)}{\partial u^s_{i,t} \partial u^g_{i,t}} & \frac{\partial^2 f_{i,t}(.,.)}{\partial (u^s_{i,t})^2} \end{pmatrix} = \begin{pmatrix} -(\rho_{i,t} b_{i,t} + 2d_{i,t}) & -\rho_{i,t} b_{i,t} \\ -\rho_{i,t} b_{i,t} & -\rho_{i,t} b_{i,t} \end{pmatrix}$$

The eigenvalues of $\nabla^2 f_{i,t}(\vec{u}_{i,t}, \vec{x}_{i,t})$ are the solutions to the characteristic equation of $\nabla^2 f_{i,t}(\vec{u}_{i,t}, \vec{x}_{i,t})$, given by $det(\nabla^2 f_{i,t}(\vec{u}_{i,t}, \vec{x}_{i,t}) - \lambda I) = 0$. Evaluating the characteristic equation, we get $\lambda^2 + 2(\rho_{i,t} b_{i,t} + d_{i,t})\lambda + 2\rho_{i,t} b_{i,t} d_{i,t} = 0$. Solving for λ in the last equation gives two distinct negative eigenvalues given by $\lambda_1 = -[(\rho_{i,t} b_{i,t} + d_{i,t}) + \sqrt{(\rho_{i,t} b_{i,t})^2 + d^2_{i,t}}]$ and $\lambda_2 = [-(\rho_{i,t} b_{i,t} + d_{i,t}) + \sqrt{(\rho_{i,t} b_{i,t})^2 + d^2_{i,t}}]$. Since the eigenvalues are negative, $\nabla^2 f_{i,t}(\vec{u}_{i,t}, \vec{x}_{i,t})$ is negative definite for all $\vec{u_{i,t}}$ and $\vec{x_{i,t}}$, and, consequently, $f_{i,t}(\vec{u}_{i,t}, \vec{x}_{i,t})$ is strictly jointly concave in $\vec{u}_{i,t}$, $i = 1, 2,\ t = 1, 2$. $\Box$

Proof of Proposition 5.2

Note that $\partial f_{i,t}(\vec{u}_{i,t}, \vec{x}_{i,t})/\partial u^g_{i,t} = \rho_{i,t} a_{i,t} - \rho_{i,t} b_{i,t}(u^g_{i,t} + u^s_{i,t}) - c_{i,t}(x^g_{i,0} - x^g_{i,t}) - 2d_{i,t} u^g_{i,t}$ and

$\partial f_{i,t}(\vec{u}_{i,t}, \vec{x}_{i,t})/\partial u^s_{i,t} = \rho_{i,t} a_{i,t} + 0.5h_{i,t} - \rho_{i,t} b_{i,t}(u^g_{i,t} + u^s_{i,t})$.

Let $u^{g*}_{i,t}$ and $u^{s*}_{i,t}$ be the values of $u^g_{i,t}$ and $u^s_{i,t}$ that solve $\partial f_{i,t}(.,.)/\partial u^g_{i,t} = 0$ and $\partial f_{i,t}(.,.)/\partial u^s_{i,t} = 0$, respectively; $u^{g*}_{i,t} = \frac{\rho_{i,t} a_{i,t} - \rho_{i,t} b_{i,t} u^s_{i,t} - c_{i,t}(x^g_{i,0} - x^g_{i,t})}{\rho_{i,t} b_{i,t} + 2d_{i,t}}$ and $u^{s*}_{i,t} = \frac{\rho_{i,t} a_{i,t} + 0.5h_{i,t} - \rho_{i,t} b_{i,t} u^g_{i,t}}{\rho_{i,t} b_{i,t}}$.

Full depletion of water in period t is guaranteed by having $u^{g*}_{i,t} > x^g_{i,t}$ and $u^{s*}_{i,t} > x^s_{i,t}$. Namely, $u^{g*}_{i,t} > x^g_{i,t}$ holds if $\rho_{i,t} a_{i,t} > (\rho_{i,t} b_{i,t} + 2d_{i,t} - c_{i,t}) x^g_{i,t} + \rho_{i,t} b_{i,t} u^s_{i,t} + c_{i,t} x^g_{i,0}$. But, we know that $x^g_{i,t} \leq x^g_{i,0}$ and $u^s_{i,t} \leq x^s_{i,0}$, where $x^g_{i,0}$ and $x^s_{i,0}$ are the initial stock levels of ground and surface water, respectively.

Hence, the last inequality holds if $\rho_{i,t} a_{i,t} > [\rho_{i,t} b_{i,t} + 2d_{i,t} - c_{i,t}]^+ x^g_{i,0} + \rho_{i,t} b_{i,t} x^s_{i,0} + c_{i,t} x^{g(\dagger)}_{i,0}$ holds, where $[\rho_{i,t} b_{i,t} + 2d_{i,t} - c_{i,t}]^+ > 0$ if $(\rho_{i,t} b_{i,t} + 2d_{i,t} - c_{i,t}) > 0$ and is zero otherwise.

Similarly, $u^{s*}_{i,t} > x^s_{i,t}$ holds if

$\rho_{i,t}a_{i,t} > \rho_{i,t}b_{i,t}(u^g_{i,t}+x^s_{i,t})-0.5h_{i,t}$ holds. Again, since $u^g_{i,t} \le x^g_{i,0}$ and $x^s_{i,t} \le x^s_{i,0}$, the last inequality holds if $\rho_{i,t}a_{i,t} > \rho_{i,t}b_{i,t}(x^g_{i,0}+x^s_{i,0})-0.5h^{(\ddagger)}_{i,t}$ holds. Combining the two inequalities in (†) and (‡) together gives $\rho_{i,t}a_{i,t} > max\{[\rho_{i,t}b_{i,t}+2d_{i,t}-c_{i,t}]^+x^g_{i,0}+\rho_{i,t}b_{i,t}x^s_{i,0}+c_{i,t}x^g_{i,0}, \rho_{i,t}b_{i,t}(x^g_{i,0}+x^s_{i,0})-0.5h_{i,t}\}$.

Consequently, under the above condition, $f_{i,t}(\vec{u}_{i,t},\vec{x}_{i,t})$ attains its maximum at $\vec{u}^*_{i,t}=\vec{x}_{i,t}$, $i=1,2,\ t=1,2$. □

Proof of Proposition 5.3

Recall that $\Gamma_{i,1}(\vec{u}_1,\vec{x}_1)=f_{i,1}(\vec{u}_{i,1},\vec{x}_{i,1})+\beta_i f^*_{i,2}(\vec{x}_{i,2},\vec{x}_{i,2})$. By Proposition 3.1, $f_{i,1}(\vec{u}_{i,1},\vec{x}_{i,1})$ is strictly jointly concave in $\vec{u}_{i,1}$. Hence, $\Gamma_{i,1}(\vec{u}_1,\vec{x}_1)$ is jointly concave in $\vec{u}_1$ if $f^*_{i,2}(\vec{x}_{i,2},\vec{x}_{i,2})$ is jointly concave in $\vec{u}_1$ because the sum of two jointly concave functions is jointly concave. Below, we show the joint concavity of $f^*_{i,2}(\vec{x}_{i,2},\vec{x}_{i,2})$ in $\vec{u}_1$. More formally, we need to show that the Hessian matrix corresponding to $f^*_{i,2}(\vec{x}_{i,2},\vec{x}_{i,2})$ is negative semi-definite (*i.e.* having non-positive eigenvalues). Note that the Hessian matrix; $\nabla^2 f^*_{i,2}(\vec{x}_{i,2},\vec{x}_{i,2})$, is given by

$$\nabla^2 f^*_{i,2}(\vec{x}_{i,2},\vec{x}_{i,2}) = \begin{pmatrix} \frac{\partial^2 f^*_{i,2}(.,.)}{\partial(u^g_{i,1})^2} & \frac{\partial^2 f^*_{i,2}(.,.)}{\partial u^g_{i,1}\partial u^s_{i,1}} & \frac{\partial^2 f^*_{i,2}(.,.)}{\partial u^g_{i,1}\partial u^g_{j,1}} & \frac{\partial^2 f^*_{i,2}(.,.)}{\partial u^g_{i,1}\partial u^s_{j,1}} \\ \frac{\partial^2 f^*_{i,2}(.,.)}{\partial u^s_{i,1}\partial u^g_{i,1}} & \frac{\partial^2 f^*_{i,2}(.,.)}{\partial(u^s_{i,1})^2} & \frac{\partial^2 f^*_{i,2}(.,.)}{\partial u^s_{i,1}\partial u^g_{j,1}} & \frac{\partial^2 f^*_{i,2}(.,.)}{\partial u^s_{j,1}\partial u^s_{j,1}} \\ \frac{\partial^2 f^*_{i,2}(.,.)}{\partial u^g_{j,1}\partial u^g_{i,1}} & \frac{\partial^2 f^*_{i,2}(.,.)}{\partial u^g_{j,1}\partial u^s_{i,1}} & \frac{\partial^2 f^*_{i,2}(.,.)}{\partial(u^g_{j,1})^2} & \frac{\partial^2 f^*_{i,2}(.,.)}{\partial u^g_{j,1}\partial u^s_{j,1}} \\ \frac{\partial^2 f^*_{i,2}(.,.)}{\partial u^s_{j,1}\partial u^g_{i,1}} & \frac{\partial^2 f^*_{i,2}(.,.)}{\partial u^s_{j,1}\partial u^s_{i,1}} & \frac{\partial^2 f^*_{i,2}(.,.)}{\partial u^s_{j,1}\partial u^g_{j,1}} & \frac{\partial^2 f^*_{i,2}(.,.)}{\partial(u^s_{j,1})^2} \end{pmatrix}$$

$$= \begin{pmatrix} (1-\alpha)^2(e_{i,2}-\rho_{i,2}b_{i,2}) & -(1-\alpha)\rho_{i,2}b_{i,2} & \alpha(1-\alpha)(e_{i,2}-\rho_{i,2}b_{i,2}) & 0 \\ -(1-\alpha)\rho_{i,2}b_{i,2} & -\rho_{i,2}b_{i,2} & -\alpha\rho_{i,2}b_{i,2} & 0 \\ \alpha(1-\alpha)(e_{i,2}-\rho_{i,2}b_{i,2}) & -\alpha\rho_{i,2}b_{i,2} & \alpha^2(e_{i,2}-\rho_{i,2}b_{i,2}) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where $e_{i,2}=2(c_{i,2}-d_{i,2})$ and $i,j=1,2,\ i\neq j$. We find that the characteristic equation of $\nabla^2 f^*_{i,2}(\vec{x}_{i,2},\vec{x}_{i,2})$ is given by $\lambda^2(\lambda^2-\sigma_1\lambda-\sigma_2)=0$, where

$\sigma_1=((1-\alpha)^2+\alpha^2)(e_{i,2}-\rho_{i,2}b_{i,2})-\rho_{i,2}b_{i,2}$ and $\sigma_2=((1-\alpha)^2+\alpha^2)\rho_{i,2}b_{i,2}e_{i,2}-(1-\alpha)^3(\rho_{i,2}b_{i,2})^2$.

Therefore, we observe that $\nabla^2 f^*_{i,2}(\vec{x}_{i,2}, \vec{x}_{i,2})$ has four eigenvalues given by $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 0.5[\sigma_1 + \sqrt{\sigma_1^2 + 4\sigma_2}]$ and $\lambda_4 = 0.5[\sigma_1 - \sqrt{\sigma_1^2 + 4\sigma_2}]$.

Now, for λ_3 and λ_4 to be non-positive, they should be real in the first place. They are real if $\sigma_1^2 + 4\sigma_2 \geq 0$. Below, we show that λ_3 and λ_4 are indeed real. Note that

$\sigma_1^2 + 4\sigma_2 = [((1-\alpha)^2+\alpha^2)(e_{i,2}-\rho_{i,2}b_{i,2})]^2 - 2[(1-\alpha)^2+\alpha^2](e_{i,2}-\rho_{i,2}b_{i,2})\rho_{i,2}b_{i,2} + (\rho_{i,2}b_{i,2})^2 + 4[(1-\alpha)^2+\alpha^2]\rho_{i,2}b_{i,2}e_{i,2} - 4(1-\alpha)^3(\rho_{i,2}b_{i,2})^2$

$= [(1-\alpha)^2+\alpha^2]^2[e_{i,2}^2 - 2\rho_{i,2}b_{i,2}e_{i,2} + (\rho_{i,2}b_{i,2})^2] + 2[(1-\alpha)^2+\alpha^2]\rho_{i,2}b_{i,2}e_{i,2} + 2[(1-\alpha)^2+\alpha^2](\rho_{i,2}b_{i,2})^2 + [1-4(1-\alpha)^3](\rho_{i,2}b_{i,2})^2 = [(1-\alpha)^2+\alpha^2]^2 e_{i,2}^2 + 4\alpha(1-\alpha)[(1-\alpha)^2+\alpha^2]\rho_{i,2}b_{i,2}e_{i,2} + \tilde{\alpha}(\rho_{i,2}b_{i,2})^2$,

where $\tilde{\alpha} = [((1-\alpha)^2+\alpha^2)(2+(1-\alpha)^2+\alpha^2) + 1 - 4(1-\alpha)^3]$.

Hence, if $\tilde{\alpha} \geq 0$, then $\sigma_1^2 + 4\sigma_2 \geq 0$ since the first two components of $(\sigma_1^2 + 4\sigma_2)$ are non-negative, for any $\alpha \in [0, 0.5]$. It is not hard to show that, with some basic algebraic simplifications, we get $\tilde{\alpha} = 4\alpha[1-\alpha^2(1-\alpha)] \geq 0$ for any $\alpha \in [0, 0.5]$. Therefore, $\sigma_1^2 + 4\sigma_2 \geq 0$, implying that λ_3 and λ_4 are real. Now, we need to show that $\lambda_3,\ \lambda_4 \leq 0$. Note that λ_3 and λ_4 are non-positive if and only if $\sigma_1, \sigma_2 \leq 0$. We also have $\sigma_1 \leq 0$ if and only if $((1-\alpha)^2+\alpha^2)[2(c_{i,2}-d_{i,2}) - \rho_{i,2}b_{i,2}] - \rho_{i,2}b_{i,2} \leq 0$, *i.e.* if and only if $2(c_{i,2}-d_{i,2}) \leq [\frac{2(1-\alpha+\alpha^2)}{(1-\alpha)^2+\alpha^2}]\rho_{i,2}b_{i,2}$. Similarly, $\sigma_2 \leq 0$ if and only if $2((1-\alpha)^2+\alpha^2)\rho_{i,2}b_{i,2}(c_{i,2}-d_{i,2}) - (1-\alpha)^3(\rho_{i,2}b_{i,2})^2 \leq 0$, *i.e.* if and only if $2(c_{i,2}-d_{i,2}) \leq [\frac{(1-\alpha)^3}{(1-\alpha)^2+\alpha^2}]\rho_{i,2}b_{i,2}$. Combining the last two inequalities together yields the condition $2(c_{i,2}-d_{i,2}) \leq min\{[\frac{2(1-\alpha+\alpha^2)}{(1-\alpha)^2+\alpha^2}]\rho_{i,2}b_{i,2}, [\frac{(1-\alpha)^3}{(1-\alpha)^2+\alpha^2}]\rho_{i,2}b_{i,2}\} = [\frac{(1-\alpha)^3}{(1-\alpha)^2+\alpha^2}]\rho_{i,2}b_{i,2}$ because $(1-\alpha)^3 \leq 2(1-\alpha+\alpha^2)$, for $\alpha \in [0, 0.5]$. Hence, if $\rho_{i,2}b_{i,2} \geq 2[\frac{(1-\alpha)^2+\alpha^2}{(1-\alpha)^3}](c_{i,2}-d_{i,2})$, all the eigenvalues are non-positive, which proves that $f^*_{i,2}(\vec{x}_{i,2}, \vec{x}_{i,2})$ is jointly concave in $\vec{u}_1$, $i = 1, 2$. $\square$

Proof of Proposition 5.5

The proof is based on the behavior of $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$ w.r.t. groundwater usage; $u^g_{i,1}$. In both (i) and (ii) of part (a), k_1 represents the first derivative of $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$ w.r.t. $u^g_{i,1}$ and k_2 represents the first derivative of $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$ w.r.t. $u^g_{i,1}$ evaluated at $u^g_{i,1} = 0$. To guarantee feasibility of the solution, we need $u^{s*}_{i,1} \leq x^s_1$, or

$w_{i,1}^* - u_{i,1}^{g*} \le x_1^s$, which gives $u_{i,1}^{g*} \ge w_{i,1}^* - x_1^s$, to hold all the time. This condition gives a lower bound on the optimal value of $u_{i,1}^{g*}$ and it must hold all the time for all solutions.

$(a)(i)$ Here, we consider the case where $\Gamma_i(\vec{u}_1, \vec{x}_1)$ is concave in $u_{i,1}^g$ ($i.e.$ $k_1 < 0$), as depicted in Figure 7.1 (a). Now, if $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$ is increasing at $u_{i,1}^g = 0$ ($i.e.$ $k_2 > 0$, as shown in the lower part of Figure 7.1 (a)), then it is optimal for user i to use as much groundwater as possible in period 1. Since $\Gamma_i(\vec{u}_1, \vec{x}_1)$ is concave and $k_2 > 0$, $\Gamma_i(\vec{u}_1, \vec{x}_1)$ attains its maximum at $\hat{u}_{i,1}^g$ where $\hat{u}_{i,1}^g$ is found from solving $\partial\Gamma_i(.,.)/\partial u_{i,1}^g = 0$ and it is given by $\hat{u}_{i,1}^g = -(k_2/k_1)$. Depending on the value of $w_{i,1}^* - x_1^s$ compared to that of $\hat{u}_{i,1}^g$, we have two cases. If $w_{i,1}^* - x_1^s < \hat{u}_{i,1}^g$, as along as we need to use as much groundwater as possible, the optimal usage policy is given by $u_{i,1}^{g*} = max\{w_{i,1}^* - x_1^s, \hat{u}_{i,1}^g\}$ and $u_{i,1}^{s*} = w_{i,1}^* - u_{i,1}^{g*}$. However, if $w_{i,1}^* - x_1^s > \hat{u}_{i,1}^g$, again since we need to use as much groundwater as possible, $w_{i,1}^* - x_1^s \le min\{w_{i,1}^*, x_1^g\}$ and, hence, the optimal usage policy is given by $u_{i,1}^{g*} = min\{x_1^g, w_{i,1}^*\}$ and $u_{i,1}^{s*} = w_{i,1}^* - u_{i,1}^{g*}$. On the other hand, if $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$ is decreasing at $u_{i,1}^g = 0$ ($i.e.$ $k_2 \le 0$, as shown in the upper part of Figure 7.1 (a)), then it is optimal for user i to use as little groundwater as possible in period 1. However, the minimum quantity of groundwater in period 1 equals to the lower bound on $u_{i,1}^{g*}$. Hence, the optimal water usage policy in this case is given by $u_{i,1}^{g*} = w_{i,1}^* - x_1^s$ and $u_{i,1}^{s*} = x_1^s$.

$(a)(ii)$ Here, we consider the case where $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$ is convex in $u_{i,1}^g$ ($i.e.$ $k_1 > 0$), as depicted in Figure 7.1 (b). Similar to (i), if $\Gamma_i(\vec{u}_1, \vec{x}_1)$ is increasing at $u_{i,1}^g = 0$ ($i.e.$ $k_2 > 0$, as shown in the upper part of Figure 7.1 (b)), then it is optimal for user i to use as much groundwater as possible in period 1. Consequently, $w_{i,1}^* - x_1^s \le min\{w_{i,1}^*, x_1^g\}$ and, hence, the optimal usage policy is given by $u_{i,1}^{g*} = min\{x_1^g, w_{i,1}^*\}$ and $u_{i,1}^{s*} = w_{i,1}^* - u_{i,1}^{g*}$. Now, if $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$ is decreasing at $u_{i,1}^g = 0$ ($i.e.$ $k \le 0$, as shown in the lower part of Figure 7.1 (b)), then it is optimal for user i to use as little groundwater as possible in period 1. Since $\Gamma_i(\vec{u}_1, \vec{x}_1)$ is convex and $k_2 \le 0$, $\Gamma_i(\vec{u}_1, \vec{x}_1)$ attains its minimum at $\hat{u}_{i,1}^g$ where $\hat{u}_{i,1}^g$ is found from solving $\partial\Gamma_i(.,.)/\partial u_{i,1}^g = 0$ and it is given by $\tilde{u}_{i,1}^g = -(k_2/k_1)$. Depending on the value of $w_{i,1}^* - x_1^s$

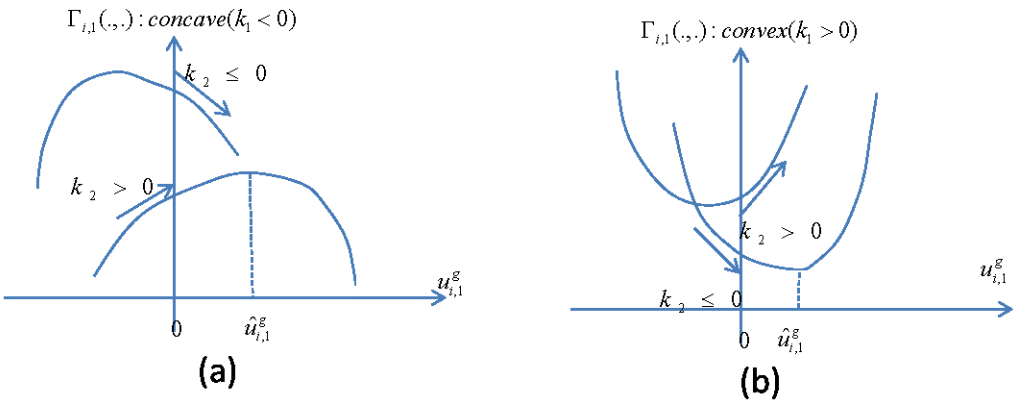


Figure 7.1: Total discounted profit vs. groundwater usage: decentralized problem

compared to that of $\hat{u}^g_{i,1}$, we have two cases. If $w^*_{i,1} - x^s_1 < \hat{u}^g_{i,1}$, since we need to use as little groundwater as possible, the optimal usage policy is given by $u^{g*}_{i,1} = max\{0, w^*_{i,1} - x^s_1\}$ and $u^{s*}_{i,1} = w^*_{i,1} - u^{g*}_{i,1}$. However, if $w^*_{i,1} - x^s_1 > \hat{u}^g_{i,1}$, again since we need to use as little groundwater as possible, $w^*_{i,1} - x^s_1 \le min\{w^*_{i,1}, x^g_1\}$ and, hence, the optimal usage policy is given by $u^{g*}_{i,1} = min\{x^g_1, w^*_{i,1}\}$ and $u^{s*}_{i,1} = w^*_{i,1} - u^{g*}_{i,1}$.

(b) If $w^*_{i,1} > x^g_1 + x^s_1$, then the optimal water usage quantities will be limited by the maximum water stock levels in period 1. Hence, the optimal water usage policy belongs to the set of water usage bounds given by $(u^{g*}_{i,1}, u^{s*}_{i,1}) \in \{(0,0), (x^g_1, 0), (0, x^s_1), (x^g_1, x^s_1)\}$. $\square$

Proof of Proposition 5.6

The solutions stated above are obtained from the usage policies given in Proposition 5.5. The optimal total water usage in period 1; $w^*_{i,1}$, in each case is obtained from optimizing the objective function, which is given by $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1) = R_i(\vec{u}_1, \vec{x}_1) - C_i(\vec{u}_1, \vec{x}_1)$. More specifically,

$$\Gamma_{i,1}(\vec{u}_1, \vec{x}_1) = [\rho_{i,1} a_{i,1}(u^g_{i,1} + u^s_{i,1}) - 0.5\rho_{i,1} b_{i,1}(u^g_{i,1} + u^s_{i,1})^2 + \beta_i \rho_{i,2} a_{i,2}(x^g_1 + x^s_1 - (1 - \alpha)u^g_{i,1} - \alpha u^g_{j,1} - u^s_{i,1}) - 0.5\beta_i \rho_{i,2} b_{i,2}(x^g_1 + x^s_1 - (1-\alpha)u^g_{i,1} - \alpha u^g_{j,1} - u^s_{i,1})^2] -$$

$[d_{i,1}(u^g_{i,1})^2 + 0.5h_{i,1}(2x^s_1 - u^s_{i,1}) + \beta_i c_{i,2}((1-\alpha)u^g_{i,1} + \alpha u^g_{j,1})(x^g_1 - (1-\alpha)u^g_{i,1} - \alpha u^g_{j,1}) + \beta_i d_{i,2}(x^g_1 - (1-\alpha)u^g_{i,1} - \alpha u^g_{j,1})^2 + 0.5\beta_i h_{i,2}(x^s_1 - u^s_{i,1})]$.

Below, for each case, we substitute the given solution in $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$ which becomes a function of $w^*_{i,1}$. Optimizing $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$ w.r.t. $w^*_{i,1}$ through setting $\partial\Gamma_{i,1}(.,.)/\partial w^*_{i,1} = 0$ and solving for $w^*_{i,1}$ gives the optimal solution as given in the proposition above.

$(a)(i)$ We substitute the solution $(u^{g*}_{i,1}, u^{s*}_{i,1}) = (x^g_1, w^*_{i,1} - x^g_1)$ in $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$ which is written as

$\Gamma_{i,1}(w^*_{i,1}) = [\rho_{i,1}a_{i,1}w^*_{i,1} - 0.5\rho_{i,1}b_{i,1}(w^*_{i,1})^2 + \beta_i\rho_{i,2}a_{i,2}(x^g_1 + x^s_1 - (1-\alpha)x^g_1 - \alpha u^g_{j,1} - w^*_{i,1} + x^g_1) - 0.5\beta_i\rho_{i,2}b_{i,2}(x^g_1 + x^s_1 - (1-\alpha)x^g_1 - \alpha u^g_{j,1} - w^*_{i,1} + x^g_1)^2] - [d_{i,1}(x^g_1)^2 + 0.5h_{i,1}(2x^s_1 - w^*_{i,1} + x^g_1) + \beta_i c_{i,2}((1-\alpha)x^g_1 + \alpha u^g_{j,1})(x^g_1 - (1-\alpha)x^g_1 - \alpha u^g_{j,1}) + \beta_i d_{i,2}(x^g_1 - (1-\alpha)x^g_1 - \alpha u^g_{j,1})^2 + 0.5\beta_i h_{i,2}(x^s_1 - w^*_{i,1} + x^g_1)]$.

Setting $\partial\Gamma_{i,1}(w^*_{i,1})/\partial w^*_{i,1} = -[\rho_{i,1}b_{i,1} + \beta_i\rho_{i,2}b_{i,2}]w^*_{i,1} + \rho_{i,1}a_{i,1} - \beta_i\rho_{i,2}a_{i,2} + \beta_i\rho_{i,2}b_{i,2}((1+\alpha)x^g_1 + x^s_1 - \alpha u^g_{j,1}) + 0.5(h_{i,1} + \beta_i h_{i,2}) = 0$, yields the solution stated in part $(a)(i)$. We observe that $w^*_{i,1}$ is a global maximizer for $\Gamma_{i,1}(w^*_{i,1})$ since $\partial^2\Gamma_{i,1}(w^*_{i,1})/\partial(w^*_{i,1})^2 = -[\rho_{i,1}b_{i,1} + \beta_i\rho_{i,2}b_{i,2}] < 0$, (*i.e.* $\Gamma_{i,1}(w^*_{i,1})$ is concave in $w^*_{i,1}$).

$(a)(ii)$ We substitute the solution $(u^{g*}_{i,1}, u^{s*}_{i,1}) = (0, w^*_{i,1})$ in $\Gamma_{i,1}(\vec{u}_1, \vec{x}_1)$ to obtain

$\Gamma_{i,1}(w^*_{i,1}) = [\rho_{i,1}a_{i,1}w^*_{i,1} - 0.5\rho_{i,1}b_{i,1}(w^*_{i,1})^2 + \beta_i\rho_{i,2}a_{i,2}(x^g_1 + x^s_1 - \alpha u^g_{j,1} - w^*_{i,1}) - 0.5\beta_i\rho_{i,2}b_{i,2}(x^g_1 + x^s_1 - \alpha u^g_{j,1} - w^*_{i,1})^2] - [0.5h_{i,1}(2x^s_1 - w^*_{i,1}) + \beta_i c_{i,2}(\alpha u^g_{j,1})(x^g_1 - \alpha u^g_{j,1}) + \beta_i d_{i,2}(x^g_1 - \alpha u^g_{j,1})^2 + 0.5\beta_i h_{i,2}(x^s_1 - w^*_{i,1})]$.

Setting $\partial\Gamma_{i,1}(w^*_{i,1})/\partial w^*_{i,1} = -[\rho_{i,1}b_{i,1} + \beta_i\rho_{i,2}b_{i,2}]w^*_{i,1} + \rho_{i,1}a_{i,1} - \beta_i\rho_{i,2}a_{i,2} + \beta_i\rho_{i,2}b_{i,2}(x^g_1 + x^s_1 - \alpha u^g_{j,1}) + 0.5(h_{i,1} + \beta_i h_{i,2}) = 0$, results in the solution stated in part $(a)(ii)$. The concavity of $\Gamma_{i,1}(w^*_{i,1})$ in $w^*_{i,1}$ is established similar to part $(a)(i)$.

$(a)(iii)$ Similarly, by letting $(u^{g*}_{i,1}, u^{s*}_{i,1}) = (\hat{u}^g_{i,1}, w^*_{i,1} - \hat{u}^g_{i,1})$, we have

$\Gamma_{i,1}(w^*_{i,1}) = [\rho_{i,1}a_{i,1}w^*_{i,1} - 0.5\rho_{i,1}b_{i,1}(w^*_{i,1})^2 + \beta_i\rho_{i,2}a_{i,2}(x^g_1 + x^s_1 - (1-\alpha)\hat{u}^g_{i,1} - \alpha u^g_{j,1} - w^*_{i,1} + \hat{u}^g_{i,1}) - 0.5\beta_i\rho_{i,2}b_{i,2}(x^g_1 + x^s_1 - (1-\alpha)\hat{u}^g_{i,1} - \alpha u^g_{j,1} - w^*_{i,1} + \hat{u}^g_{i,1})^2] -$

$[d_{i,1}(\hat{u}^g_{i,1})^2 + 0.5h_{i,1}(2x^s_1 - w^*_{i,1} + \hat{u}^g_{i,1}) + \beta_i c_{i,2}((1-\alpha)\hat{u}^g_{i,1} + \alpha u^g_{j,1})(x^g_1 - (1-\alpha)\hat{u}^g_{i,1} - \alpha u^g_{j,1}) + \beta_i d_{i,2}(x^g_1 - (1-\alpha)\hat{u}^g_{i,1} - \alpha u^g_{j,1})^2 + 0.5\beta_i h_{i,2}(x^s_1 - w^*_{i,1} + \hat{u}^g_{i,1})]$.

Note from $\hat{u}^g_{i,1}$ in Proposition 5.5, we can rewrite $\hat{u}^g_{i,1}$ as $\hat{u}^g_{i,1} = \gamma_0 + \gamma_1 w^*_{i,1}$, where γ_0 and γ_1 are as defined before. Moreover, we have $\frac{\partial \hat{u}^g_{i,1}}{\partial w^*_{i,1}} = \frac{\beta_i \alpha \rho_{i,2} b_{i,2}}{k_1}$. Then, by deriving $\Gamma_{i,1}(w^*_{i,1})$ w.r.t. $w^*_{i,1}$ and simplifying algebraically, we get

$\partial\Gamma_{i,1}(w^*_{i,1})/\partial w^*_{i,1} = -k_1\rho_{i,1}a_{i,1} + k_1\rho_{i,1}b_{i,1}w^*_{i,1} + \beta_i\rho_{i,2}b_{i,2}(\beta_i\alpha^2\rho_{i,2}b_{i,2} + k_1) - \beta_i\rho_{i,2}b_{i,2}(\beta_i\alpha^2\rho_{i,2}b_{i,2} + k_1)(x^g_1 + x^s_1 - \alpha\hat{u}^g_{i,1} - \alpha u^g_{j,1} - w^*_{i,1}) - 2\beta_i\alpha\rho_{i,2}b_{i,2}d_{i,1}\hat{u}^g_{i,1} - 0.5h_{i,1}(\beta_i\alpha\rho_{i,2}b_{i,2} + k_1) - (\beta_i)^2\alpha(1-\alpha)c_{i,2}\rho_{i,2}b_{i,2}(x^g_1 - (1-\alpha)\hat{u}^g_{i,1} - \alpha u^g_{j,1}) + (\beta_i)^2\alpha(1-\alpha)c_{i,2}\rho_{i,2}b_{i,2}((1-\alpha)\hat{u}^g_{i,1} + \alpha u^g_{j,1}) + 2\beta_i\alpha(1-\alpha)d_{i,2}\rho_{i,2}b_{i,2}(x^g_1 - (1-\alpha)\hat{u}^g_{i,1} - \alpha u^g_{j,1}) - 0.5\beta_i h_{i,2}(\beta_i\alpha\rho_{i,2}b_{i,2} + k_1) = 0$

Substituting $\hat{u}^g_{i,1} = \gamma_0 + \gamma_1 w^*_{i,1}$ in the above equation and doing some algebraic simplifications gives

$[k_1\rho_{i,1}b_{i,1} - \beta_i(\beta_i\alpha^2\rho_{i,2}b_{i,2} + k_1)\rho_{i,2}b_{i,2}(\alpha\gamma_1 - 1) - 2\beta_i\alpha\gamma_1\rho_{i,2}b_{i,2}d_{i,1} + 2(\beta_i)^2\alpha(1-\alpha)^2\gamma_1\rho_{i,2}b_{i,2}d_{i,1}(c_{i,2} - d_{i,2})]w^*_{i,1} = [k_1\rho_{i,1}a_{i,1} + \beta_i\rho_{i,2}b_{i,2}(\beta_i\alpha^2\rho_{i,2}b_{i,2} + k_1)[x^g_1 + x^s_1 - \alpha u^g_{j,1} - \alpha\gamma_0 - 1] + 2\beta_i\alpha\gamma_0\rho_{i,2}b_{i,2}d_{i,1} + 0.5(h_{i,1} + \beta_i h_{i,2})(\beta_i\alpha\rho_{i,2}b_{i,2} + k_1) + \beta_i^2\alpha(1-\alpha)\rho_{i,2}b_{i,2}(c_{i,2} - 2d_{i,2})(x^g_1 - \alpha u^g_{j,1}) - 2\beta_i^2\alpha(1-\alpha)^2\rho_{i,2}b_{i,2}c_{i,2}(c_{i,2} - d_{i,2})\gamma_0 - \beta_i^2\alpha^2(1-\alpha)\rho_{i,2}b_{i,2}c_{i,2}u^g_{j,1}$.

The last equation, can be written as $w^*_{i,1} = \tilde{k_2}/\tilde{k_1}$, where

$\tilde{k_1} = k_1\rho_{i,1}b_{i,1} - \beta_i(\beta_i\alpha^2\rho_{i,2}b_{i,2} + k_1)\rho_{i,2}b_{i,2}(\alpha\gamma_1 - 1) - 2\beta_i\alpha\gamma_1\rho_{i,2}b_{i,2}d_{i,1} + 2(\beta_i)^2\alpha(1-\alpha)^2\gamma_1\rho_{i,2}b_{i,2}d_{i,1}(c_{i,2} - d_{i,2})$,

$\tilde{k_2} = k_1\rho_{i,1}a_{i,1} + \beta_i\rho_{i,2}b_{i,2}(\beta_i\alpha^2\rho_{i,2}b_{i,2} + k_1)[x^g_1 + x^s_1 - \alpha u^g_{j,1} - \alpha\gamma_0 - 1] + 2\beta_i\alpha\gamma_0\rho_{i,2}b_{i,2}d_{i,1} + 0.5(h_{i,1} + \beta_i h_{i,2})(\beta_i\alpha\rho_{i,2}b_{i,2} + k_1) + \beta_i^2\alpha(1-\alpha)\rho_{i,2}b_{i,2}(c_{i,2} - 2d_{i,2})(x^g_1 - \alpha u^g_{j,1}) - 2\beta_i^2\alpha(1-\alpha)^2\rho_{i,2}b_{i,2}c_{i,2}(c_{i,2} - d_{i,2})\gamma_0 - \beta_i^2\alpha^2(1-\alpha)\rho_{i,2}b_{i,2}c_{i,2}u^g_{j,1}$.

Again, the concavity of $\Gamma_{i,1}(w^*_{i,1})$ in $w^*_{i,1}$ is established similar to part $(a)(i)$. Namely, it is concave if $\tilde{k_1} < 0$.

$(a)(iv)$ For $(u^{g*}_{i,1}, u^{s*}_{i,1}) = (w^*_{i,1}, 0)$, we have

$\Gamma_{i,1}(w^*_{i,1}) = [\rho_{i,1}a_{i,1}w^*_{i,1} - 0.5\rho_{i,1}b_{i,1}(w^*_{i,1})^2 + \beta_i\rho_{i,2}a_{i,2}(x^g_1 + x^s_1 - (1-\alpha)w^*_{i,1} - \alpha u^g_{j,1}) - 0.5\beta_i\rho_{i,2}b_{i,2}(x^g_1 + x^s_1 - (1-\alpha)w^*_{i,1} - \alpha u^g_{j,1})^2] - [d_{i,1}(w^*_{i,1})^2 + 0.5h_{i,1}(2x^s_1) + \beta_i c_{i,2}((1-\alpha)w^*_{i,1} + \alpha u^g_{j,1})(x^g_1 - (1-\alpha)w^*_{i,1} - \alpha u^g_{j,1}) + \beta_i d_{i,2}(x^g_1 - (1-\alpha)w^*_{i,1} - \alpha u^g_{j,1})^2 + 0.5\beta_i h_{i,2}(x^s_1)]$, and

$\partial\Gamma_{i,1}(w^*_{i,1})/\partial w^*_{i,1} = \rho_{i,1}a_{i,1} - \rho_{i,1}b_{i,1}w^*_{i,1} - \beta_i(1-\alpha)\rho_{i,2}a_{i,2} + \beta_i(1-\alpha)\rho_{i,2}b_{i,2}(x^g_1 + x^s_1 - (1-\alpha)w^*_{i,1} - \alpha u^g_{j,1}) - 2d_{i,1}w^*_{i,1} + \beta_i(1-\alpha)c_{i,2}((1-\alpha)w^*_{i,1} + \alpha u^g_{j,1}) - \beta_i(1-\alpha)c_{i,2}(x^g_1 - (1-\alpha)w^*_{i,1} - \alpha u^g_{j,1}) + 2\beta_i(1-\alpha)d_{i,2}(x^g_1 - (1-\alpha)w^*_{i,1} - \alpha u^g_{j,1}) = 0$. Solving for $w^*_{i,1}$, we obtain the solution given in part $(a)(iv)$. The solution $w^*_{i,1}$ is a global maximizer of $\Gamma_{i,1}(w^*_{i,1})$ since $\partial^2\Gamma_{i,1}(w^*_{i,1})/\partial(w^*_{i,1})^2 = -[\rho_{i,1}b_{i,1} + 2d_{i,1} + \beta_i(1-\alpha)^2\rho_{i,2}b_{i,2} + 2\beta_i(1-\alpha)^2(c_{i,2} - d_{i,2})] < 0$.

$(a)(v)$ Letting $(u^{g*}_{i,1}, u^{s*}_{i,1}) = (w^*_{i,1} - x^s_1, x^s_1)$ gives

$\Gamma_{i,1}(w^*_{i,1}) = [\rho_{i,1}a_{i,1}w^*_{i,1} - 0.5\rho_{i,1}b_{i,1}(w^*_{i,1})^2 + \beta_i\rho_{i,2}a_{i,2}(x^g_1 + (1-\alpha)x^s_1 - (1-\alpha)w^*_{i,1} - \alpha u^g_{j,1}) - 0.5\beta_i\rho_{i,2}b_{i,2}(x^g_1 + (1-\alpha)x^s_1 - (1-\alpha)w^*_{i,1} - \alpha u^g_{j,1})^2] - [d_{i,1}(w^*_{i,1} - x^s_1)^2 + 0.5h_{i,1}(x^s_1) + \beta_i c_{i,2}((1-\alpha)w^*_{i,1} - (1-\alpha)x^s_1 + \alpha u^g_{j,1})(x^g_1 + (1-\alpha)x^s_1 - (1-\alpha)w^*_{i,1} - \alpha u^g_{j,1}) + \beta_i d_{i,2}(x^g_1 + (1-\alpha)x^s_1 - (1-\alpha)w^*_{i,1} - \alpha u^g_{j,1})^2 + 0.5\beta_i h_{i,2}(0)]$, and

$\partial\Gamma_{i,1}(w^*_{i,1})/\partial w^*_{i,1} = \rho_{i,1}a_{i,1} - \rho_{i,1}b_{i,1}w^*_{i,1} - \beta_i(1-\alpha)\rho_{i,2}a_{i,2} + \beta_i(1-\alpha)\rho_{i,2}b_{i,2}(x^g_1 + (1-\alpha)x^s_1 - (1-\alpha)w^*_{i,1} - \alpha u^g_{j,1}) - 2d_{i,1}(w^*_{i,1} - x^s_1) + \beta_i(1-\alpha)c_{i,2}((1-\alpha)w^*_{i,1} - (1-\alpha)x^s_1 + \alpha u^g_{j,1}) - \beta_i(1-\alpha)c_{i,2}(x^g_1 + (1-\alpha)x^s_1 - (1-\alpha)w^*_{i,1} - \alpha u^g_{j,1}) + 2\beta_i(1-\alpha)d_{i,2}(x^g_1 + (1-\alpha)x^s_1 - (1-\alpha)w^*_{i,1} - \alpha u^g_{j,1}) = 0$. Solving for $w^*_{i,1}$ yields the solution given in part $(a)(v)$. The concavity of $\Gamma_{i,1}(w^*_{i,1})$ in $w^*_{i,1}$ is established similar to part $(a)(iv)$.

(b) In part (a), when we evaluate each optimal solution, if $w^*_{i,1}$ happens to be infeasible, then the corresponding solution should be dropped from the candidate optimal solutions set. However, there is at least one feasible and optimal solution given by the initial stock level bounds given by $(u^{g*}_{i,1}, u^{s*}_{i,1}) \in \{(0,0), (x^g_1, 0), (0, x^s_1), (x^g_1, x^s_1)\}$, $i = 1, 2$. $\square$

Proof of Corollary 5.1

Under the identical users case, we substitute $\rho_{i,t} = \rho_t,\ a_{i,t} = a_t,\ b_{i,t} = b_t,\ c_{i,t} = c_t,\ d_{i,t} = d_t$ and $\beta_i = \beta$ Accordingly, $(u^{g*}_{i,1}, u^{s*}_{i,1}) = (u^{g*}_1, u^{s*}_1)$ and, hence, $w^*_{i,1} = w^*_1$, for all i. In the sequel, for each solution in Proposition 5.6, the response values of user j will be substituted by $u^g_{j,1} = u^{g*}_{i,1}$ and $u^s_{j,1} = u^{s*}_{i,1}$, implying that both users have the same optimal solution. In all cases, the concavity of $\Gamma_{i,1}(w^*_1)$ w.r.t. w^*_1 is established similar to that in the Proposition 5.6.

(i) We substitute $u^g_{j,1} = x^g_1$ in part (a)(i) of Proposition 5.6 to get the above solution in part (i). Similarly, in part (a)(ii) of Proposition 5.6, we substitute $u^g_{j,1} = 0$. It turns out that the solution is the same as the solution in the previous part.

(ii) When $u^g_{j,1} = \hat{u}^g_1$, under the identical setting, it is easy to show that $\hat{u}^g_1$ reduces to $\hat{u}^g_1 = \epsilon_0 + \epsilon_1 w^*_1$, where $\epsilon_0 = \frac{\beta\alpha[\rho_2 a_2 - \rho_2 b_2(x^g_1 + x^s_1)] - 0.5(h_1 + \beta h_2) - \beta(1-\alpha)(c_2 - 2d_2)x^g_1}{2d_1 + 2\beta(1-\alpha)(d_2 - c_2)}$ and $\epsilon_1 = \frac{\beta\rho_2 b_2}{2d_1 + 2\beta(1-\alpha)(d_2 - c_2)}$.

Substituting $\hat{u}^g_1$ in $\Gamma_{i,1}(w^*_1)$ and optimizing w.r.t. w^*_1 yields the following $w^*_1 = \frac{\rho_1 a_1 - \beta(1-\alpha)\rho_2 a_2 + \beta(1-\alpha)\rho_2 b_2(x^g_1 + x^s_1) - \beta(1-\alpha)(c_2 - 2d_2)x^g_1}{\rho_1 b_1 + \beta(1-\alpha)\rho_2 b_2 + 2d_1 - 2\beta(1-\alpha)(c_2 - d_2)}$.

(iii) Substituting $u^g_{j,1} = w^*_1$ in part (a)(iv) of Proposition 5.6 and doing some algebraic simplifications yields the solution stated above in part (a)(ii).

(iv) We substitute $u^g_{j,1} = w^*_1 - x^s_1$ in part (a)(v) of Proposition 5.6 and simplify more to get the solution stated above in part (a)(iii).

(b) Similar to the proof of part (b) in Proposition 5.6 and, hence, omitted. $\square$

Proof of Proposition 5.7

The proof is based on the behavior of $\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1)$ w.r.t. groundwater usage; $u^g_{i,1}$. In both (i) and (ii) of part (a), k_1 represents the first derivative of $\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1)$ w.r.t. $u^g_{i,1}$ and k_2 represents the first derivative of $\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1)$ w.r.t. $u^g_{i,1}$ evaluated at $u^g_{i,1} = 0$. To guarantee feasibility of the solution, we need $u^{s*}_{i,1} \leq x^s_1$, or $w^*_{i,1} - u^{g*}_{i,1} \leq x^s_1$, which gives $u^{g*}_{i,1} \geq w^*_{i,1} - x^s_1$, to hold all the time. This condition gives a lower bound on the optimal value of $u^{g*}_{i,1}$ and it must hold all the time for all solutions.

(a)(i) Here, we consider the case where $\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1)$ is concave in $u^g_{i,1}$ (*i.e.* $k_1 < 0$), as depicted in Figure 7.2 (a). Now, if $\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1)$ is increasing at $u^g_{i,1} = 0$ (*i.e.* $k_2 > 0$, as shown in the lower part of Figure 7.2 (a)), then it is optimal for user i to use as much groundwater as possible in period 1. Since $\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1)$ is concave and $k_2 > 0$, $\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1)$ attains its maximum at $\hat{u}^g_{i,1}$ where $\hat{u}^g_{i,1}$ is found from solving $\partial\tilde{\Gamma}_1(.,.)/\partial u^g_{i,1} = 0$ and it is given by $\hat{u}^g_{i,1} = -(k_2/k_1)$. Depending on the value of $w^*_{i,1} - x^s_1$ compared to that of $\hat{u}^g_{i,1}$, we have two cases. If $w^*_{i,1} - x^s_1 < \hat{u}^g_{i,1}$, as along as we need to use as much groundwater as possible, the optimal usage policy is given by $u^{g*}_{i,1} = max\{w^*_{i,1} - x^s_1, \hat{u}^g_{i,1}\}$ and $u^{s*}_{i,1} = w^*_{i,1} - u^{g*}_{i,1}$. However, if $w^*_{i,1} - x^s_1 > \hat{u}^g_{i,1}$, again since we need to use as much groundwater as possible, $w^*_{i,1} - x^s_1 \leq min\{w^*_{i,1}, x^g_1\}$ and, hence, the optimal usage policy is given by $u^{g*}_{i,1} = min\{x^g_1, w^*_{i,1}\}$ and $u^{s*}_{i,1} = w^*_{i,1} - u^{g*}_{i,1}$. On the other hand, if $\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1)$ is decreasing at $u^g_{i,1} = 0$ ($i.e.$ $k_2 \leq 0$, as shown in the upper part of Figure 7.2 (a)), then it is optimal for user i to use as little groundwater as possible in period 1. However, the minimum quantity of groundwater in period 1 equals to the lower bound on $u^{g*}_{i,1}$. Hence, the optimal water usage policy in this case is given by $u^{g*}_{i,1} = w^*_{i,1} - x^s_1$ and $u^{s*}_{i,1} = x^s_1$.

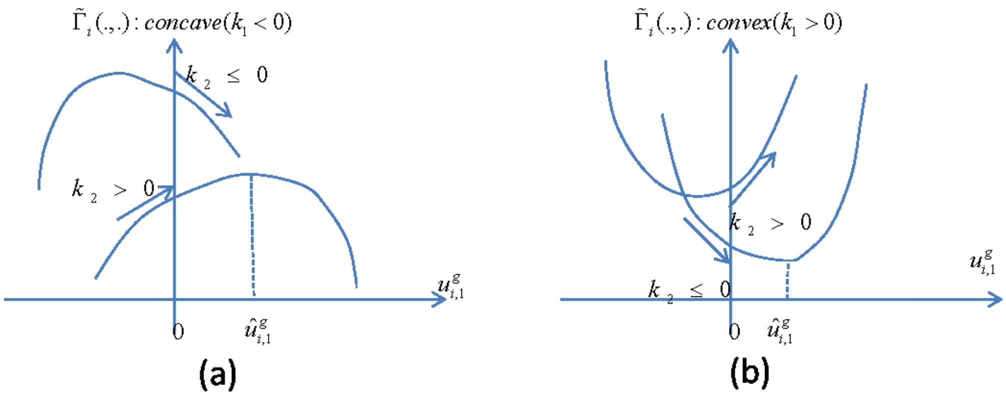


Figure 7.2: Total discounted profit vs. groundwater usage: centralized problem

$(a)(ii)$ Here, we consider the case where $\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1)$ is convex in $u^g_{i,1}$ ($i.e.$ $k_1 > 0$), as depicted in Figure 7.2 (b). Similar to (i), if $\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1)$ is increasing at $u^g_{i,1} = 0$ ($i.e.$ $k_2 > 0$, as shown in the upper part of Figure 7.2 (b)), then it is optimal for user i to use as much groundwater as possible in period 1. Consequently, $w^*_{i,1} - x^s_1 \leq min\{w^*_{i,1}, x^g_1\}$ and, hence, the optimal usage policy is given by $u^{g*}_{i,1} = min\{x^g_1, w^*_{i,1}\}$ and $u^{s*}_{i,1} = w^*_{i,1} - u^{g*}_{i,1}$. Now, if $\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1)$ is decreasing at $u^g_{i,1} = 0$ ($i.e.$ $k \leq 0$, as shown in the lower part of Figure 7.2 (b)), then it is optimal for user i to use as little groundwater as possible in period 1. Since $\tilde{\Gamma_1}(\vec{u}_1, \vec{x}_1)$ is convex and $k_2 \leq 0$, $\tilde{\Gamma_1}(\vec{u}_1, \vec{x}_1)$ attains its minimum at $\hat{u}^g_{i,1}$ where $\hat{u}^g_{i,1}$ is found from solving $\partial\tilde{\Gamma_1}(.,.)/\partial u^g_{i,1} = 0$ and it is given by $\tilde{u}^g_{i,1} = -(k_2/k_1)$. Depending on the value of $w^*_{i,1} - x^s_1$ compared to that of $\hat{u}^g_{i,1}$, we have two cases. If $w^*_{i,1} - x^s_1 < \hat{u}^g_{i,1}$, since we need to use as little groundwater as possible, the optimal usage policy is given by $u^{g*}_{i,1} = max\{0, w^*_{i,1} - x^s_1\}$ and $u^{s*}_{i,1} = w^*_{i,1} - u^{g*}_{i,1}$. However, if $w^*_{i,1} - x^s_1 > \hat{u}^g_{i,1}$, again since we need to use as little groundwater as possible, $w^*_{i,1} - x^s_1 \leq min\{w^*_{i,1}, x^g_1\}$ and, hence, the optimal usage policy is given by $u^{g*}_{i,1} = min\{x^g_1, w^*_{i,1}\}$ and $u^{s*}_{i,1} = w^*_{i,1} - u^{g*}_{i,1}$.

(*b*) If $w^*_{i,1} > x^g_1 + x^s_1$, then the optimal water usage quantities will be limited by the maximum water stock levels in period 1. Hence, the optimal water usage policy belongs to the set of water usage bounds given by $(u^{g*}_{i,1}, u^{s*}_{i,1}) \in \{(0,0), (x^g_1, 0), (0, x^s_1), (x^g_1, x^s_1)\}$. $\square$

Proof of Proposition 5.8

The solutions stated above are obtained from the usage policies given in Proposition 5.7. The optimal total water usage in period 1; $w^*_{i,1}$, in each case is obtained from optimizing the objective function, which is given by $\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1) = \tilde{R}_1(\vec{u}_1, \vec{x}_1) - \tilde{C}_1(\vec{u}_1, \vec{x}_1)$. In particular, we have

$$\begin{aligned}\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1) = {} & \rho_{i,1}a_{i,1}(u^g_{i,1} + u^s_{i,1}) - 0.5\rho_{i,1}b_{i,1}(u^g_{i,1} + u^s_{i,1})^2 + \rho_{j,1}a_{j,1}(u^g_{j,1} + u^s_{j,1}) - \\ & 0.5\rho_{j,1}b_{j,1}(u^g_{j,1} + u^s_{j,1})^2 + \beta[\rho_{i,2}a_{i,2}(x^g_1 + x^s_1 - (1-\alpha)u^g_{i,1} - \alpha u^g_{j,1} - u^s_{i,1}) - \\ & 0.5\rho_{i,2}b_{i,2}(x^g_1 + x^s_1 - (1-\alpha)u^g_{i,1} - \alpha u^g_{j,1} - u^s_{i,1})^2] + \beta[\rho_{j,2}a_{j,2}(x^g_1 + x^s_1 - (1- \\ & \alpha)u^g_{j,1} - \alpha u^g_{i,1} - u^s_{j,1}) - 0.5\rho_{j,2}b_{j,2}(x^g_1 + x^s_1 - (1-\alpha)u^g_{j,1} - \alpha u^g_{i,1} - u^s_{j,1})^2] - \\ & [d_{i,1}(u^g_{i,1})^2 + 0.5h_{i,1}(2x^s_1 - u^s_{i,1})] - [d_{j,1}(u^g_{j,1})^2 + 0.5h_{j,1}(2x^s_1 - u^s_{j,1})] - \beta[c_{i,2}((1- \\ & \alpha)u^g_{i,1} + \alpha u^g_{j,1})(x^g_1 - (1-\alpha)u^g_{i,1} - \alpha u^g_{j,1}) + d_{i,2}(x^g_1 - (1-\alpha)u^g_{i,1} - \alpha u^g_{j,1})^2 + \\ & 0.5h_{i,2}(x^s_1 - u^s_{i,1})] - \beta[c_{j,2}((1-\alpha)u^g_{j,1} + \alpha u^g_{i,1})(x^g_1 - (1-\alpha)u^g_{j,1} - \alpha u^g_{i,1}) + \\ & d_{j,2}(x^g_1 - (1-\alpha)u^g_{j,1} - \alpha u^g_{i,1})^2 + 0.5h_{j,2}(x^s_1 - u^s_{j,1})].\end{aligned}$$

Below, for each case, we substitute the given solution in $\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1)$ which becomes a function of $w^*_{i,1}$. Optimizing $\tilde{\Gamma}_1(\vec{u}_1, \vec{x}_1)$ w.r.t. $w^*_{i,1}$ through setting

$\partial\tilde{\Gamma}_1(\vec{u}_1,\vec{x}_1)/\partial w^*_{i,1} = 0$ and solving for $w^*_{i,1}$ gives the optimal solution as given in the proposition above.

$(a)(i)$ We substitute the solution $(u^{g*}_{i,1}, u^{s*}_{i,1}) = (x^g_1, w^*_{i,1} - x^g_1)$ in $\tilde{\Gamma}_1(\vec{u}_1,\vec{x}_1)$ which is written as

$\tilde{\Gamma}_1(w^*_{i,1}) = \rho_{i,1}a_{i,1}w^*_{i,1} - 0.5\rho_{i,1}b_{i,1}(w^*_{i,1})^2 + \rho_{j,1}a_{j,1}(u^g_{j,1}+u^s_{j,1}) - 0.5\rho_{j,1}b_{j,1}(u^g_{j,1}+u^s_{j,1})^2 + \beta[\rho_{i,2}a_{i,2}(x^g_1+x^s_1-(1-\alpha)x^g_1-\alpha u^g_{j,1}-w^*_{i,1}+x^g_1) - 0.5\rho_{i,2}b_{i,2}(x^g_1+x^s_1-(1-\alpha)x^g_1-\alpha u^g_{j,1}-w^*_{i,1}+x^g_1)^2] + \beta[\rho_{j,2}a_{j,2}(x^g_1-\alpha x^g_1-(1-\alpha)u^g_{j,1}+x^s_1-u^s_{j,1}) - 0.5\rho_{i,2}b_{i,2}(x^g_1-\alpha x^g_1-(1-\alpha)u^g_{j,1}+x^s_1-u^s_{j,1})^2] - [d_{i,1}(x^g_1)^2+0.5h_{i,1}(2x^s_1-w^*_{i,1}+x^g_1)] - [d_{j,1}(u^g_{j,1})^2+0.5h_{j,1}(2x^s_1-u^s_{j,1})] - \beta[c_{i,2}((1-\alpha)x^g_1+\alpha u^g_{j,1})(x^g_1-(1-\alpha)x^g_1-\alpha u^g_{j,1}) + d_{i,2}(x^g_1-(1-\alpha)x^g_1-\alpha u^g_{j,1})^2 + 0.5h_{i,2}(x^s_1-w^*_{i,1}+x^g_1)] - \beta[c_{j,2}(\alpha x^g_1+(1-\alpha)u^g_{j,1})(x^g_1-\alpha x^g_1-(1-\alpha)u^g_{j,1}) + d_{j,2}(x^g_1-\alpha x^g_1-(1-\alpha)u^g_{j,1})^2 + 0.5h_{j,2}(x^s_1-u^g_{j,1})]$.

Setting $\partial\tilde{\Gamma}_1(w^*_{i,1})/\partial w^*_{i,1} = \rho_{i,1}a_{i,1} - \rho_{i,1}b_{i,1}w^*_{i,1} - \beta\rho_{i,2}a_{i,2} + \beta\rho_{i,2}b_{i,2}((1+\alpha)x^g_1 + x^s_1 - \alpha u^g_{j,1} - w^*_{i,1}) + 0.5(h_{i,1}+\beta h_{i,2}) = 0$, from which gives the solution stated in part $(a)(i)$. We observe that $w^*_{i,1}$ is a global maximizer for $\tilde{\Gamma}_1(w^*_{i,1})$ since $\partial^2\tilde{\Gamma}_1(w^*_{i,1})/\partial(w^*_{i,1})^2 = -[\rho_{i,1}b_{i,1} + \beta\rho_{i,2}b_{i,2}] < 0$, (*i.e.* $\tilde{\Gamma}_1(w^*_{i,1})$ is concave in $w^*_{i,1}$).

$(a)(ii)$ We substitute the solution $(u^{g*}_{i,1}, u^{s*}_{i,1}) = (0, w^*_{i,1})$ in$\tilde{\Gamma}_1(\vec{u}_1,\vec{x}_1)$ to obtain

$\tilde{\Gamma}_1(w^*_{i,1}) = \rho_{i,1}a_{i,1}w^*_{i,1} - 0.5\rho_{i,1}b_{i,1}(w^*_{i,1})^2 + \rho_{j,1}a_{j,1}(u^g_{j,1}+u^s_{j,1}) - 0.5\rho_{j,1}b_{j,1}(u^g_{j,1}+u^s_{j,1})^2 + \beta[\rho_{i,2}a_{i,2}(x^g_1+x^s_1-\alpha u^g_{j,1}-w^*_{i,1}) - 0.5\rho_{i,2}b_{i,2}(x^g_1+x^s_1-\alpha u^g_{j,1}-w^*_{i,1})^2] + \beta[\rho_{j,2}a_{j,2}(x^g_1-(1-\alpha)u^g_{j,1}+x^s_1-u^s_{j,1}) - 0.5\rho_{i,2}b_{i,2}(x^g_1-(1-\alpha)u^g_{j,1}+x^s_1-u^s_{j,1})^2] - [d_{i,1}(0)^2+0.5h_{i,1}(2x^s_1-w^*_{i,1})] - [d_{j,1}(u^g_{j,1})^2+0.5h_{j,1}(2x^s_1-u^s_{j,1})] - \beta[c_{i,2}(\alpha u^g_{j,1})(x^g_1-\alpha u^g_{j,1}) + d_{i,2}(x^g_1-\alpha u^g_{j,1})^2 + 0.5h_{i,2}(x^s_1-w^*_{i,1})] - \beta[c_{j,2}((1-\alpha)u^g_{j,1})(x^g_1-(1-\alpha)u^g_{j,1}) + d_{j,2}(x^g_1-(1-\alpha)u^g_{j,1})^2 + 0.5h_{j,2}(x^s_1-u^g_{j,1})]$.

We have $\partial\Gamma_{i,1}(w^*_{i,1})/\partial w^*_{i,1} = \rho_{i,1}a_{i,1} - \rho_{i,1}b_{i,1}w^*_{i,1} - \beta\rho_{i,2}a_{i,2} + \beta\rho_{i,2}b_{i,2}(x^g_1 + x^s_1 - \alpha u^g_{j,1} - w^*_{i,1}) + 0.5(h_{i,1}+\beta h_{i,2}) = 0$, which results in the solution stated in part $(a)(ii)$. The concavity of $\partial\tilde{\Gamma}_1(w^*_{i,1})$ in $w^*_{i,1}$ is established similar to part $(a)(i)$.

$(a)(iii)$ Similarly, letting $(u^{g*}_{i,1}, u^{s*}_{i,1}) = (\hat{u}^g_{i,1}, w^*_{i,1} - \hat{u}^g_{i,1})$ gives

$\tilde{\Gamma}_1(w^*_{i,1}) = \rho_{i,1}a_{i,1}w^*_{i,1} - 0.5\rho_{i,1}b_{i,1}(w^*_{i,1})^2 + \rho_{j,1}a_{j,1}(u^g_{j,1}+u^s_{j,1}) - 0.5\rho_{j,1}b_{j,1}(u^g_{j,1}+u^s_{j,1})^2 + \beta[\rho_{i,2}a_{i,2}(x^g_1+x^s_1-(1-\alpha)\hat{u}^g_{i,1}-\alpha u^g_{j,1}-w^*_{i,1}+\hat{u}^g_{i,1}) - 0.5\rho_{i,2}b_{i,2}(x^g_1+$

$x_1^s - (1-\alpha)\hat{u}_{i,1}^g - \alpha u_{j,1}^g - w_{i,1}^* + \hat{u}_{i,1}^g)^2] + \beta[\rho_{j,2}a_{j,2}(x_1^g - \alpha\hat{u}_{i,1}^g - (1-\alpha)u_{j,1}^g + x_1^s - u_{j,1}^s) - 0.5\rho_{i,2}b_{i,2}(x_1^g - \alpha\hat{u}_{i,1}^g - (1-\alpha)u_{j,1}^g + x_1^s - u_{j,1}^s)^2] - [d_{i,1}(\hat{u}_{i,1}^g)^2 + 0.5h_{i,1}(2x_1^s - w_{i,1}^* + \hat{u}_{i,1}^g)] - [d_{j,1}(u_{j,1}^g)^2 + 0.5h_{j,1}(2x_1^s - u_{j,1}^s)] - \beta[c_{i,2}((1-\alpha)\hat{u}_{i,1}^g + \alpha u_{j,1}^g)(x_1^g - (1-\alpha)\hat{u}_{i,1}^g - \alpha u_{j,1}^g) + d_{i,2}(x_1^g - (1-\alpha)\hat{u}_{i,1}^g - \alpha u_{j,1}^g)^2 + 0.5h_{i,2}(x_1^s - w_{i,1}^* + \hat{u}_{i,1}^g)] - \beta[c_{j,2}(\alpha\hat{u}_{i,1}^g + (1-\alpha)u_{j,1}^g)(x_1^g - \alpha\hat{u}_{i,1}^g - (1-\alpha)u_{j,1}^g) + d_{j,2}(x_1^g - \alpha\hat{u}_{i,1}^g - (1-\alpha)u_{j,1}^g)^2 + 0.5h_{j,2}(x_1^s - u_{j,1}^g)]$.

Note from $\hat{u}_{i,1}^g$ in Proposition 5.7, we can rewrite $\hat{u}_{i,1}^g$ as $\hat{u}_{i,1}^g = \tilde{\gamma}_0 + \tilde{\gamma}_1 w_{i,1}^*$,, where $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ are as defined before. Substituting $\hat{u}_{i,1}^g$ in $\tilde{\Gamma}_1(w_{i,1}^*)$ given above, deriving w.r.t. $w_{i,1}^*$ and simplifying algebraically, we get

$[\rho_{i,1}b_{i,1} + \beta(\alpha\tilde{\gamma}_1 - 1)^2\rho_{i,2}b_{i,2} + \beta(\alpha\tilde{\gamma}_1)^2\rho_{j,2}b_{j,2} + 2(\tilde{\gamma}_1)^2 d_{i,1} + 2\beta(1-\alpha\tilde{\gamma}_1)^2(d_{i,2} - c_{i,2}) + 2\beta(\alpha\tilde{\gamma}_1)^2(d_{j,2} - c_{j,2})]w_{i,1}^* = \tilde{k_2} = \rho_{i,1}a_{i,1} + \beta(\alpha\tilde{\gamma}_1 - 1)\rho_{i,2}a_{i,2} - \beta\alpha\tilde{\gamma}_1\rho_{j,2}a_{j,2} - \beta(\alpha\tilde{\gamma}_1 - 1)\rho_{i,2}b_{i,2}(x_1^g + x_1^s + \alpha\tilde{\gamma}_0) + \beta\alpha\tilde{\gamma}_1\rho_{j,2}b_{j,2}(x_1^g + x_1^s - w_{j,1} - \alpha\tilde{\gamma}_0 + \alpha u_{j,1}^g) - 2\tilde{\gamma}_0\tilde{\gamma}_1 d_{i,1} - 0.5(\tilde{\gamma}_1 - 1)(h_{i,1} + \beta h_{i,2} + \beta(1-\alpha)\tilde{\gamma}_1 c_{i,2}((1-\alpha)\tilde{\gamma}_0 + \alpha u_{j,1}^g) + \beta\alpha\tilde{\gamma}_1 c_{j,2}(\alpha\tilde{\gamma}_0 + (1-\alpha)u_{j,1}^g) - \beta(1-\alpha)\tilde{\gamma}_1(c_{i,2} - 2d_{i,2})(x_1^g - (1-\alpha)\tilde{\gamma}_0 - \alpha u_{j,1}^g) - \beta\alpha\tilde{\gamma}_1(c_{j,2} - 2d_{j,2})(x_1^g - \alpha\tilde{\gamma}_0 - (1-\alpha)u_{j,1}^g)$,

Again, the concavity of $\Gamma_{i,1}(w_{i,1}^*)$ in $w_{i,1}^*$ is established similar to part $(a)(i)$. Namely, it is concave if $\tilde{k_1} < 0$.

$(a)(iv)$ For $(u_{i,1}^{g*}, u_{i,1}^{s*}) = (w_{i,1}^*, 0)$, we obtain

$\tilde{\Gamma}_1(w_{i,1}^*) = \rho_{i,1}a_{i,1}w_{i,1}^* - 0.5\rho_{i,1}b_{i,1}(w_{i,1}^*)^2 + \rho_{j,1}a_{j,1}(u_{j,1}^g + u_{j,1}^s) - 0.5\rho_{j,1}b_{j,1}(u_{j,1}^g + u_{j,1}^s)^2 + \beta[\rho_{i,2}a_{i,2}(x_1^g + x_1^s - (1-\alpha)w_{i,1}^* - \alpha u_{j,1}^g) - 0.5\rho_{i,2}b_{i,2}(x_1^g + x_1^s - (1-\alpha)w_{i,1}^* - \alpha u_{j,1}^g)^2] + \beta[\rho_{j,2}a_{j,2}(x_1^g - \alpha w_{i,1}^* - (1-\alpha)u_{j,1}^g + x_1^s - u_{j,1}^s) - 0.5\rho_{i,2}b_{i,2}(x_1^g - \alpha w_{i,1}^* - (1-\alpha)u_{j,1}^g + x_1^s - u_{j,1}^s)^2] - [d_{i,1}(w_{i,1}^*)^2 + 0.5h_{i,1}(2x_1^s)] - [d_{j,1}(u_{j,1}^g)^2 + 0.5h_{j,1}(2x_1^s - u_{j,1}^s)] - \beta[c_{i,2}((1-\alpha)w_{i,1}^* + \alpha u_{j,1}^g)(x_1^g - (1-\alpha)w_{i,1}^* - \alpha u_{j,1}^g) + d_{i,2}(x_1^g - (1-\alpha)w_{i,1}^* - \alpha u_{j,1}^g)^2 + 0.5h_{i,2}(x_1^s)] - \beta[c_{j,2}(\alpha w_{i,1}^* + (1-\alpha)u_{j,1}^g)(x_1^g - \alpha w_{i,1}^* - (1-\alpha)u_{j,1}^g) + d_{j,2}(x_1^g - \alpha w_{i,1}^* - (1-\alpha)u_{j,1}^g)^2 + 0.5h_{j,2}(x_1^s - u_{j,1}^g)]$, and

$\partial\tilde{\Gamma}_1(w_{i,1}^*)/\partial w_{i,1}^* = \rho_{i,1}a_{i,1} - \rho_{i,1}b_{i,1}w_{i,1}^* + \beta[-(1-\alpha)\rho_{i,2}a_{i,2} + (1-\alpha)\rho_{i,2}b_{i,2}(x_1^g + x_1^s - (1-\alpha)w_{i,1}^* - \alpha u_{j,1}^g)] + \beta[-\alpha\rho_{j,2}a_{j,2} + \alpha\rho_{j,2}b_{j,2}(x_1^g + x_1^s - \alpha w_{i,1}^* - (1-\alpha)u_{j,1}^g - u_{j,1}^s)] - 2d_{i,1}w_{i,1}^* - \beta[(1-\alpha)c_{i,2}((1-\alpha)w_{i,1}^* + \alpha u_{j,1}^g) + (1-\alpha)c_{i,2}(x_1^g - (1-\alpha)w_{i,1}^* - \alpha u_{j,1}^g) - 2(1-\alpha)d_{i,2}(x_1^g - (1-\alpha)w_{i,1}^* - \alpha u_{j,1}^g)] - \beta[-\alpha c_{j,2}(\alpha w_{i,1}^* + (1-\alpha)u_{j,1}^g) + \alpha c_{j,2}(x_1^g - \alpha w_{i,1}^* - (1-\alpha)u_{j,1}^g) - 2\alpha d_{j,2}(x_1^g - \alpha w_{i,1}^* - (1-\alpha)u_{j,1}^g)] = 0.$

or,

$\partial\tilde{\Gamma}_1(w^*_{i,1})/\partial w^*_{i,1} = -[\rho_{i,1}b_{i,1} + \beta[(1-\alpha)^2\rho_{i,2}b_{i,2} + \alpha^2\rho_{j,2}b_{j,2}] + 2d_{i,1} - 2\beta[(1-\alpha)^2c_{i,2} + \alpha^2c_{j,2}] + 2\beta[(1-\alpha)^2d_{i,2} + \alpha^2d_{j,2}]]w^*_{i,1} + \rho_{i,1}a_{i,1} - \beta[(1-\alpha)\rho_{i,2}a_{i,2} + \alpha\rho_{j,2}a_{j,2}] + \beta(1-\alpha)\rho_{i,2}b_{i,2}(x^g_1 + x^s_1 - \alpha u^g_{j,1}) + \beta\alpha\rho_{j,2}b_{j,2}(x^g_1 + x^s_1 - (1-\alpha)u^g_{j,1} - u^s_{j,1}) + \beta(1-\alpha)\alpha u^g_{j,1}(c_{i,2} + c_{j,2}) - \beta(1-\alpha)c_{i,2}(x^g_1 - \alpha u^g_{j,1}) - \beta\alpha c_{j,2}(x^g_1 - (1-\alpha)u^g_{j,1}) + 2\beta(1-\alpha)d_{i,2}(x^g_1 - \alpha u^g_{j,1}) - \beta\alpha d_{j,2}(x^g_1 - (1-\alpha)u^g_{j,1}) = 0.$

which is written as $-\gamma w^*_{i,1} = \theta$, where γ and θ are as given above. Solving for $w^*_{i,1}$, we obtain the solution given in part $(a)(iv)$. However, observe that $\partial^2\tilde{\Gamma}_1(w^*_{i,1})/\partial(w^*_{i,1})^2 = -[\rho_{i,1}b_{i,1} + \beta[(1-\alpha)^2\rho_{i,2}b_{i,2} + \alpha^2\rho_{j,2}b_{j,2}] + 2d_{i,1} - 2\beta[(1-\alpha)^2c_{i,2} + \alpha^2c_{j,2}] + 2\beta[(1-\alpha)^2d_{i,2} + \alpha^2d_{j,2}]]$. The given solution $w^*_{i,1}$ is a global maximizer of $\tilde{\Gamma}_1(w^*_{i,1})$ since $\partial^2\tilde{\Gamma}_1(w^*_{i,1})/\partial(w^*_{i,1})^2 = -[\rho_{i,1}b_{i,1} + \beta[(1-\alpha)^2\rho_{i,2}b_{i,2} + \alpha^2\rho_{j,2}b_{j,2}] + 2d_{i,1} + 2\beta(1-\alpha)^2(c_{i,2} - d_{i,2}) + 2\beta\alpha^2(c_{j,2} - d_{j,2})] < 0.$

$(a)(v)$ Letting $(u^{g*}_{i,1}, u^{s*}_{i,1}) = (w^*_{i,1} - x^s_1, x^s_1)$ results in

$\tilde{\Gamma}_1(w^*_{i,1}) = \rho_{i,1}a_{i,1}w^*_{i,1} - 0.5\rho_{i,1}b_{i,1}(w^*_{i,1})^2 + \rho_{j,1}a_{j,1}(u^g_{j,1} + u^s_{j,1}) - 0.5\rho_{j,1}b_{j,1}(u^g_{j,1} + u^s_{j,1})^2 + \beta[\rho_{i,2}a_{i,2}(x^g_1 - (1-\alpha)w^*_{i,1} + (1-\alpha)x^s_1 - \alpha u^g_{j,1}) - 0.5\rho_{i,2}b_{i,2}(x^g_1 - (1-\alpha)w^*_{i,1} + (1-\alpha)x^s_1 - \alpha u^g_{j,1})^2] + \beta[\rho_{j,2}a_{j,2}(x^g_1 - \alpha w^*_{i,1} + \alpha x^s_1 - (1-\alpha)u^g_{j,1} + x^s_1 - u^s_{j,1}) - 0.5\rho_{i,2}b_{i,2}(x^g_1 - \alpha w^*_{i,1} + \alpha x^s_1 - (1-\alpha)u^g_{j,1} + x^s_1 - u^s_{j,1})^2] - [d_{i,1}(w^*_{i,1} - x^s_1)^2 + 0.5h_{i,1}(x^s_1)] - [d_{j,1}(u^g_{j,1})^2 + 0.5h_{j,1}(2x^s_1 - u^s_{j,1})] - \beta[c_{i,2}((1-\alpha)w^*_{i,1} - (1-\alpha)x^s_1 + \alpha u^g_{j,1})(x^g_1 - (1-\alpha)w^*_{i,1} - (1-\alpha)x^s_1 - \alpha u^g_{j,1}) + d_{i,2}(x^g_1 - (1-\alpha)w^*_{i,1} - (1-\alpha)x^s_1 - \alpha u^g_{j,1})^2 + 0.5h_{i,2}(0)] - \beta[c_{j,2}(\alpha w^*_{i,1} - \alpha x^s_1 + (1-\alpha)u^g_{j,1})(x^g_1 - \alpha w^*_{i,1} + \alpha x^s_1 - (1-\alpha)u^g_{j,1}) + d_{j,2}(x^g_1 - \alpha w^*_{i,1} - \alpha x^s_1 - (1-\alpha)u^g_{j,1})^2 + 0.5h_{j,2}(x^s_1 - u^s_{j,1})],$
and

$\partial\tilde{\Gamma}_1(w^*_{i,1})/\partial w^*_{i,1} = -[\rho_{i,1}b_{i,1} + \beta[(1-\alpha)^2\rho_{i,2}b_{i,2} + \alpha^2\rho_{j,2}b_{j,2}] + 2d_{i,1} - 2\beta[(1-\alpha)^2c_{i,2} + \alpha^2c_{j,2}] + 2\beta[(1-\alpha)^2d_{i,2} + \alpha^2d_{j,2}]]w^*_{i,1} + \rho_{i,1}a_{i,1} - \beta[(1-\alpha)\rho_{i,2}a_{i,2} + \alpha\rho_{j,2}a_{j,2}] + \beta(1-\alpha)\rho_{i,2}b_{i,2}(x^g_1 + (1-\alpha)x^s_1 - \alpha u^g_{j,1}) + \beta\alpha\rho_{j,2}b_{j,2}(x^g_1 + \alpha x^s_1 + x^s_1 - (1-\alpha)u^g_{j,1} - u^s_{j,1}) + d_{i,1}x^s_1 + \beta(1-\alpha)c_{i,2}(\alpha u^g_{j,1} - (1-\alpha)x^s_1) + \beta\alpha c_{j,2}((1-\alpha)u^g_{j,1} - \alpha x^s_1) - \beta(1-\alpha)c_{i,2}(x^g_1 + (1-\alpha)x^s_1 - \alpha u^g_{j,1}) - \beta\alpha c_{j,2}(x^g_1 + \alpha x^s_1 - (1-\alpha)u^g_{j,1}) + 2\beta(1-\alpha)d_{i,2}(x^g_1 + (1-\alpha)x^s_1 - \alpha u^g_{j,1}) + 2\beta\alpha d_{j,2}(x^g_1 + \alpha x^s_1 - (1-\alpha)u^g_{j,1}) = 0$

which is written as $-\gamma w^*_{i,1} = \sigma$, where γ and σ are as given above. Solving for $w^*_{i,1}$ yields the given solution. Again, the concavity condition corresponding to this solution is established similar to that in the previous part.

(b) Similar to the proof of part (b) in Proposition 5.6 and, hence, omitted. □

Proof of Corollary 5.3

Similar to the proof in Corollary 5.1, all revenue-cost parameters are assumed to be identical, but time-variant, across users. Also, we have $(u^{g*}_{i,1}, u^{s*}_{i,1}) = (u^{g*}_1, u^{s*}_1)$ and, hence, $w^*_{i,1} = w^*_1$, for all i. Eventually, for each solution in Proposition 5.8, the solution values of user j will be given by $u^g_{j,1} = u^{g*}_{i,1}$ and $u^s_{j,1} = u^{s*}_{i,1}$.

(i) We substitute $u^g_{j,1} = x^g_1$ in part $(a)(i)$ of Proposition 5.8 to get the solution given above in part (i). Similarly, in part $(a)(ii)$ of Proposition 5.8, we substitute $u^g_{j,1} = 0$, which yields the same solution similar to that when $u^g_{j,1} = x^g_1$. Likewise, for the solution in part $(a)(iii)$, we substitute $u^g_{j,1} = \hat{u}^g_1$, which gives the same solution as well. The value of $\hat{u}^g_1$ follows immediately in accordance with the identical setting of users.

(ii) By substituting $u^g_{j,1} = w^*_1$ and $u^s_{j,1} = 0$ in the solution given in Proposition 5.8 in part $(a)(iv)$ and doing some algebraic simplifications, we get the solution stated above in part $(a)(ii)$. The respective concavity condition follows accordingly.

(iii) In this case, we substitute $u^g_{j,1} = w^*_1 - x^s_1$ and $u^s_{i,1} = x^s_1$ in the solution given in Proposition 5.8 in part $(a)(v)$ and simplify more to get the solution stated above in part $(a)(iii)$. Similarly, the concavity condition follows in accordance with the identical setting.

(iv) Similar to the proof of part (b) in Proposition 5.6 and, hence, omitted. $\square$

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